P23·9

Vectors

CHAPTER ASSIGNMENT

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. State vectors in column or unit vector form (i, j, k) as requested and give exact surds or angles to 1 decimal place.

Practice questions from *Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook* (Cambridge University Press, Sue Pemberton and Julianne Hughes), Chapter 9 (Vectors), Exercises 9A to 9E and End-of-chapter review exercise 9, for classroom use. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 9A · Displacement and translation vectors

9A.1 The diagram shows the vectors AB and BC on 1 centimetre squared paper (A to B is 5 units right, 3 units down; B to C is 6 units right, 2 units up). Using 1 cm as 1 unit of displacement: (a) write AB and BC as column vectors · (b) write down the column vector AC.

9A.2 The diagram shows the vectors DE and DF on 1 centimetre squared paper (D to E is 7 units right, 2 units up; D to F is 3 units right, 5 units up). Using 1 cm as 1 unit of displacement: (a) find the column vector EF · (b) show that DF − DE = EF.

9A.3 In triangle PQR, justify the statement QR = PR − PQ.

9A.4 In triangle ABC, X and Y are the midpoints of AB and AC, respectively. AX = a and AY = b. (a) Use a vector method to prove that BC is parallel to XY. · (b) $|XY| = k|BC|$. Write down the value of the constant k.

9A.5 The diagram shows a cuboid, ABCDEFGH. O is the origin, with x, y, z axes along OA, OD (into page) and OE (vertical). M is the midpoint of BC and the point N is on FG such that FN : NG is 1 : 3. Given that $AG = (12, 4, 2)^T$, find the displacement vector: (a) AM · (b) AN.

9A.6 ABCDEFGH is a regular octagon. AB = p, AH = q, HG = r and HC = s. (a) Using the vectors p, q, r and s, write down expressions for: (i) BC · (ii) EB. (b) Explain why s = kp and find the exact value of k.

9A.7 Use a vector method to show that, when the diagonals of a quadrilateral bisect one another, then the opposite sides are parallel and equal in length.

Exercise 9B · Position vectors and 3D geometry

9B.1 Relative to an origin O, the position vectors of the points A and B are given by OA = 5i + 3j and OB = 5i + λj + μk. (a) In the case where λ = 0 and μ = 7, find the unit vector in the direction of AB. · (b) Given that triangle OAB is right-angled with area 5√34 and μ = 10, find the value of λ.

9B.2 The diagram shows a cuboid OABCDEFG with a horizontal base OABC. The cuboid has a length OA of 8 cm, a width AB of 4 cm and a height BF of d cm. The point M is the midpoint of CE and the point N is the point on GF such that GN = 6 cm. The unit vectors i, j and k are parallel to OA, OC and OD, respectively. It is given that OM = 4i + 2j + k. (a) Write down the height of the cuboid. · (b) Find ON. · (c) Find the unit vector in the direction MN.

9B.3 The position vectors of the points A, B, C and D are given by a, b, c and d, respectively, where $a = (2, 2, 0)^T$, $b = (13, 5, 4)^T$, $c = (5, -3, 4)^T$ and $d = (-6, -6, 0)^T$. (a) (i) Find $|AD|$, $|BC|$, $|AB|$ and $|DC|$. · (ii) Deduce that ABCD is a parallelogram. (b) Find the coordinates of: (i) the point M, the midpoint of the line AB · (ii) the point P such that $|BP| = (1/3)|BD|$.

9B.4 Relative to an origin O, the position vector of A is $i - 2j + 5k$ and the position vector of B is $3i + 4j + k$. (a) Find the magnitude of AB. · (b) Use Pythagoras' theorem to show that OAB is a right-angled triangle. · (c) Find the exact area of triangle OAB.

9B.5 The position vectors of the points A, B and C are given by $a = 5i + j + 2k$, $b = 7i + 4j - k$ and $c = i + j + qk$. Given that the length of AB is equal to the length of AC, find the exact possible values of the constant q.

9B.6 The diagram shows a triangular prism, OABCDE. The uniform cross-section of the prism, OAE, is a right-angled triangle with base 24 cm and height 7 cm. The length, AB, of the prism is 20 cm. The unit vectors i, j and k are parallel to OA, OC and OE, respectively. The point N divides the length of the line DB in the ratio 2 : 3. (a) Find the magnitude of DB. · (b) Find ON.

9B.7 Three points O, P and Q are such that $OP = 2i + 5j + ak$ and $OQ = i + (1 + a)j - 3k$. Given that the magnitude of OP is equal to the magnitude of OQ, find the value of the constant a.

9B.8 Relative to an origin O, the position vectors of the points P and Q are given by $OP = (-6k, -2, 8(1 + k))^T$ and $OQ = (2k + 13, -8, -32k)^T$. Given that OPQ is a straight line: (a) find the value of the constant k · (b) write each of OP and OQ in the form $xi + yj + zk$ · (c) find the magnitude of the vector PQ.

Exercise 9C · The scalar product

9C.1 Find which of the following pairs of position vectors are perpendicular to one another. For any position vectors that are not perpendicular, find the acute angle between them: (a) $\mathbf{a} = (2, 8, -2)^{\mathrm{T}}$, $\mathbf{b} = (7, -1, 3)^{\mathrm{T}}$ · (b) $\mathbf{c} = (5, 12, 13)^{\mathrm{T}}$, $\mathbf{d} = (-3, -1, 3)^{\mathrm{T}}$ · (c) $\mathbf{e} = (4, -9, -2)^{\mathrm{T}}$, $\mathbf{f} = (-4, -2, 1)^{\mathrm{T}}$

9C.2 Relative to the origin O, the position vectors of the points A and B are given by $\overrightarrow{OA} = (-5, 0, 3)^{\mathrm{T}}$ and $\overrightarrow{OB} = (1, 7, 2)^{\mathrm{T}}$. Find angle BOA.

9C.3 Given that the vectors $6\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ and $a\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ are perpendicular, find the value of the constant a.

9C.4 Relative to the origin O, the position vectors of the points P and Q are given by OP = $(5k, -3, 7k + 9)^T$, OQ = $(k, k + 2, -1)^T$ where k is a constant. (a) Find the values of k for which OP and OQ are perpendicular to one another. · (b) Given that k = 2, find the angle, θ, between OP and OQ.

9C.5 The diagram shows a cuboid OABCDEFG in which OC = 6 cm, OA = 4 cm and OD = 3 cm. Unit vectors i, j and k are parallel to OC, OA and OD respectively. The point M lies on FB such that FM = 1 cm, N is the midpoint of OC and P is the centre of the rectangular face DEFG. Using a scalar product, find angle NPM.

9C.6 The position vector a is such that a = 4i − 8j + k. Find a · j and hence find the angle that a makes with the positive y-axis.

9C.7 Explain why $\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}$ has no meaning.

9C.8 The diagram shows a cube OABCDEFG with side length 4 units. The unit vectors i, j and k are parallel to OA, OC and OD, respectively. The point M is the midpoint of GF and AN : NB is 1 : 3. (a) Find each of the vectors OM and NG. · (b) Find the angle between the directions OM and NG.

9C.9 The diagram shows a triangular prism, OABCDE. The uniform cross-section of the prism, OAE, is a right-angled triangle with base 77 cm and height 36 cm. The length, AB, of the prism is 60 cm. The unit vectors i, j and k are parallel to OA, OC and OE, respectively. The point N divides the length of the line DB in the ratio 4 : 1 and M is the midpoint of DE. Use a scalar product to find angle MAN.

9C.10 The diagram shows a pyramid VOABC. The base of the pyramid, OABC, is a rectangle. The unit vectors i, j and k are parallel to OA, OC and OV, respectively. The position vectors of the points A, B, C and V are given by OA = 6i, OC = 3j and OV = 9k. The points M and N are the midpoints of AB and BV, respectively. (a) Find the angle between the directions of AN and OC. · (b) Find the vector MN. · (c) The point P lies on OV and is such that PNM is a right angle. Find the position vector of the point P.

Exercise 9D · The vector equation of a line

9D.1 Write down the vector equation for the line through the point A that is parallel to vector b when: (a) A(0, −1, 5) and b = 2i + 6j − k · (b) A(0, 0, 0) and b = 7i − j − k · (c) A(7, 2, −3) and b = 3i − 4k.

9D.2 Write each of your answers to question 1 in parametric form.

9D.3 Show that the line through the points with position vectors $9\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$ and $\mathbf{i} + 7\mathbf{j} + \mathbf{k}$ is parallel to the line $\mathbf{r} = \lambda(16\mathbf{i} - 10\mathbf{j} - 12\mathbf{k})$.

9D.4 Find the parametric equations of each of the lines: (a) $\mathbf{r} = 2\mathbf{i} + 13\mathbf{j} + \mathbf{k} + t(\mathbf{i} + \mathbf{j} - \mathbf{k})$ · (b) $\mathbf{r} = 10\mathbf{j} + t(2\mathbf{i} + 5\mathbf{j})$ · (c) $\mathbf{r} = \mathbf{i} - 3\mathbf{j} + t(2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k})$.

9D.5 (a) Find the parametric equations of the line L through the points A(0, 4, −2) and B(1, 1, 6). · (b) Hence find the coordinates of the point where the line L crosses the Oxy plane.

9D.6 The line L_1 has vector equation $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k} + \lambda(6\mathbf{i} + \mathbf{j} + 3\mathbf{k})$ and the line L_2 has parametric equations $x = \mu + 4$, $y = \mu - 7$, $z = 3\mu$. (a) Show that L_1 and L_2 are not parallel. · (b) Find the angle between the line L_1 and the line L_2.

9D.7 The vector equation of the line L_1 is given by $(x, y, z)^T = (-3, 0, 8)^T + t(4, -1, -3)^T$. (a) Find the vector equation of the line L_2 that is parallel to L_1 and which passes through the point A(5, −3, 2). · (b) Show that A(5, −3, 2) is the foot of the perpendicular from the point B(4, −7, 2) to the line L_2.

9D.8 The points A, B and C are position vectors relative to the origin O, given by $\mathrm{OA} = (3, 1, 5)^T$, $\mathrm{OB} = (0, -1, 2)^T$ and $\mathrm{OC} = (1, 2, 3)^T$. (a) Find a vector equation for the line L passing through A and B. · (b) The line through C, perpendicular to L, meets L at the point N. Find the exact coordinates of N.

Exercise 9E · Intersection of two lines

9E.1 Find whether the following pairs of lines are parallel, intersecting or skew. State the position vector of the common point of any lines that intersect: (a) $x = 3\lambda + 4$, $y = 5\lambda + 1$, $z = 4\lambda - 3$ and $x = \mu + 1$, $y = \mu + 4$, $z = 2\mu + 5$ · (b) $\mathbf{r} = \mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + \lambda(3\mathbf{i} - 4\mathbf{j} + 9\mathbf{k})$ and $\mathbf{r} = \mathbf{i} - 5\mathbf{j} + 4\mathbf{k} + \lambda(-6\mathbf{i} + 8\mathbf{j} - 18\mathbf{k})$ · (c) $\mathbf{r} = 7\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{j} + 9\mathbf{k})$ and $\mathbf{r} = 4\mathbf{j} + \mathbf{k} + \mu(8\mathbf{i} + \mathbf{j} + 14\mathbf{k})$ · (d) $\mathbf{r} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} + \lambda(6\mathbf{i} + 4\mathbf{j} - 2\mathbf{k})$ and $\mathbf{r} = 4\mathbf{i} - 9\mathbf{j} + 2\mathbf{k} + \mu(-\mathbf{i} - 8\mathbf{j} - \mathbf{k})$.

9E.2 Given that the lines with vector equations $\mathbf{r} = 2\mathbf{i} + 9\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} - 4\mathbf{j} + 5\mathbf{k})$ and $\mathbf{r} = 11\mathbf{i} + 9\mathbf{j} + p\mathbf{k} + \mu(-\mathbf{i} - 2\mathbf{j} + 16\mathbf{k})$ intersect at the point P, find the value of p and the position vector of the point P.

9E.3 Three lines, L_1, L_2 and L_3, have vector equations: $\mathbf{r} = 16\mathbf{i} - 4\mathbf{j} - 6\mathbf{k} + \lambda(-12\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$, $\mathbf{r} = 16\mathbf{i} + 28\mathbf{j} + 15\mathbf{k} + \mu(8\mathbf{i} + 8\mathbf{j} + 5\mathbf{k})$, and $\mathbf{r} = \mathbf{i} + 9\mathbf{j} + 3\mathbf{k} + \nu(4\mathbf{i} - 12\mathbf{j} - 8\mathbf{k})$. The 3 points of intersection of these lines form an acute-angled triangle. For this triangle, find: (a) the position vector of each of the three vertices · (b) the size of each of the interior angles · (c) the length of each side.

9E.4 The line L_1 passes through the points (3, 7, 9) and (−1, 3, 4). (a) Find a vector equation of the line L_1. · (b) The line L_2 has vector equation $\mathbf{r} = \mathbf{i} + 2\mathbf{j} + \mathbf{k} + \mu(3\mathbf{j} + 2\mathbf{k})$. Show that L_1 and L_2 do not intersect.

9E.5 The point A has coordinates (1, 0, 5) and the point B has coordinates (−1, 2, 9). (a) Find the vector AB. · (b) Write down a vector equation of the line AB. · (c) Find the acute angle between the line AB and the line L with vector equation $\mathbf{r} = \mathbf{i} + 3\mathbf{j} + 4\mathbf{k} + \mu(-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$. · (d) Find the point of intersection of the line AB and the line L.

End-of-chapter review exercise 9

R9.1 Relative to the origin O, the position vectors of points A, B and C are given by $\mathrm{OA} = (2, 3, 7)^{\mathrm{T}}$, $\mathrm{OB} = (3m, 1, 1)^{\mathrm{T}}$ and $\mathrm{OC} = (4, -m, -m(m + 1))^{\mathrm{T}}$. (a) It is given that AB = OC. Find the angle OAB. · (b) Find the vector equation of the line AC.

R9.2 With respect to the origin O, the position vectors of the points A and B are given by $\mathrm{OA} = (-2, 0, 6)^{\mathrm{T}}$ and $\mathrm{OB} = (1, -1, 4)^{\mathrm{T}}$. (a) Find whether or not the vectors OA and OB are perpendicular. · (b) (i) Write down the vector AB. (ii) Find a vector equation of the line AB. · (c) The vector equation of the line L is given by $\mathbf{r} = 7\mathbf{i} - 12\mathbf{j} + 7\mathbf{k} + \mu(-3\mathbf{i} + 10\mathbf{j} - 5\mathbf{k})$. Show that the lines AB and L intersect and find the position vector of the point of intersection.

R9.3 The diagram shows a prism, ABCDEFGH, with a parallelogram-shaped uniform cross-section. The point E is such that OE is the height of the parallelogram. The point M is such that OM is parallel to DC and N is the midpoint of DE. The side OD has a length of 5 units. The unit vectors i, j and k are parallel to OA, OM and OE, respectively. The position vectors of the points A, E, C and M are given by OA = 9i, OM = 15j and OE = 12k. (a) Express the vectors AH and NH in terms of i, j and k. · (b) Use a vector method to find angle AHN. · (c) Write down a vector equation of the line AH.

R9.4 The position vectors of A, B and C relative to an origin O are given by OA = $(6, 3, 2)^T$, OB = $(2, n, -1)^T$ and OC = $(8, 9, 0)^T$ where n is a constant. (a) Find the value of n for which $|AB| = |CB|$. · (b) In this case, use a scalar product to find angle ABC.

R9.5 (a) Given the vectors $8i - 2j + 5k$ and $i + 2j + pk$ are perpendicular, find the value of the constant p. · (b) The line L_1 passes through the point $(-3, 1, 5)$ and is parallel to the vector $7i - j - k$. (i) Write down a vector equation of the line L_1. (ii) The line L_2 has vector equation $r = i - 2j + 2k + \mu(i + 8j - 3k)$. Show that L_1 and L_2 do not intersect.

R9.6 The origin O and the points A, B and C are such that OABC is a rectangle. With respect to O, the position vectors of the points A and B are $-4\mathbf{i} + p\mathbf{j} - 6\mathbf{k}$ and $-10\mathbf{i} - 2\mathbf{j} - 10\mathbf{k}$. (a) Find the value of the positive constant p. · (b) Find a vector equation of the line AC. · (c) Show that the line AC and the line, L, with vector equation $\mathbf{r} = 3\mathbf{i} + 7\mathbf{j} + \mathbf{k} + \mu(-4\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$ intersect and find the position vector of the point of intersection. · (d) Find the acute angle between the lines AC and L.

R9.7 Relative to the origin O, the points A, B, C and D have position vectors $OA = 4\mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $OB = 2\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$, $OC = 2\mathbf{j} + 7\mathbf{k}$, $OD = -6\mathbf{i} + 22\mathbf{j} + 9\mathbf{k}$. (a) Use a scalar product to show that angle ABC is a right angle. · (b) Show that $AD = k\mathbf{BC}$, where k is a constant, and explain what this means. · (c) The point E is the midpoint of the line AD. Find a vector equation of the line EC.

R9.8 With respect to the origin O, the points A, B, C, D have position vectors given by OA = −3i + j + 8k, OB = −10i + 2j + 15k, OC = 5i − 2j − 2k, OD = i + 6j + 10k. (a) Calculate the acute angle between the lines AB and CD. · (b) Prove that the lines AB and CD intersect. · (c) The point E has position vector 5i + 3j + 4k. Show that the perpendicular distance from E to the line CD is equal to $\sqrt{5}$.

R9.9 PQRS is a rhombus. The coordinates of the vertices P, Q and S are (9, 2, 4), (−0.5, 6, 6.5) and (4.5, −4, −3.5), respectively. (a) Find the vectors PQ and PS. · (b) Find the coordinates of R. · (c) Show that PQRS is in fact a square and find the length of the side of the square. · (d) The point T is the centre of the square. Find the coordinates of T. · (e) The point V has coordinates (5, 17.5, −13.5). (i) Find a vector equation of the line VT. (ii) Verify that T is the foot of the perpendicular from V to PR. (iii) Describe the solid VPQRS.

R9.10 The points P and Q have coordinates (0, 19, −1) and (−6, 26, −11), respectively. The line L has vector equation $\mathbf{r} = 3\mathbf{i} + 9\mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} - 10\mathbf{j} + 3\mathbf{k})$. (a) Show that the point P lies on the line L. · (b) Find the magnitude of PQ. · (c) Find the obtuse angle between PQ and L. · (d) Calculate the perpendicular distance from Q to the line L.

R9.11 The points A and B have coordinates (7, 1, 6) and (10, 5, 1), respectively. (a) Write down a vector equation of the line AB. · (b) The point P lies on the line AB. The point Q has coordinates (0, −5, 7). Given that PQ is perpendicular to AB, find a vector equation of the line PQ. · (c) Find the shortest distance from Q to the line AB.

R9.12 The line L_1 has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} + \lambda(4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$. The points A(3, p, 5) and B(q, 0, 2), where p and q are constants, lie on the line L_1. (a) Find the value of p and the value of q. (b) The line L_2 has vector equation $\mathbf{r} = 3\mathbf{j} + \mathbf{k} + \mu(7\mathbf{i} + \mathbf{j} + 7\mathbf{k})$. Show that L_1 and L_2 intersect and find the position vector of the point of intersection. · (c) Find the acute angle between L_1 and L_2.

Check yourself

C9.1 The points A, B, C have position vectors OA = 2i − j + 3k, OB = 5i + 2j − k and OC = −i + 4j + 2k. Find the unit vector in the direction of AB and calculate the exact angle BAC.

C9.2 Line L_1 passes through (1, −2, 4) with direction 2i + j − 2k. Line L_2 has vector equation r = (4i + 3j + k) + μ(i + 2j + 2k). Determine whether L_1 and L_2 intersect, are parallel, or are skew. If they intersect, find their common point.

C9.3 Find the coordinates of the foot of the perpendicular from the point P(4, 1, −3) to the straight line r = (2i − j + 5k) + t(i + 2j − k), and hence calculate the shortest distance from P to the line.

C9.4 Relative to origin O, points A and B have position vectors OA = 3i + 4j + 5k and OB = i − 2j + 3k. Find the position vector of point C on line AB such that AC : CB = 3 : 2.

C9.5 The line L has equation r = (2i + j − k) + λ(3i − 4j + 12k). Point Q has position vector 8i + 9j + 2k. (a) Show that the parameter value at the foot of the perpendicular N from Q to L is λ = 1. (b) Hence find the exact length of QN.

Chapter quiz

Q1 Find the magnitude of the vector v = 2i − 3j + 6k.

A	B	C	D
7	√11	49	5

Q2 Which vector is a unit vector in the direction of a = 4i − 4j + 2k?

A	B	C	D
(1/6)(4i − 4j + 2k)	(1/36)(4i − 4j + 2k)	(1/10)(4i − 4j + 2k)	(1/6)(2i − 2j + k)

Q3 Find the displacement vector AB from A(3, −1, 4) to B(1, 5, −2).

A	B	C	D
−2i + 6j − 6k	2i − 6j + 6k	4i + 4j + 2k	−2i + 4j − 6k

Q4 For what value of p are vectors a = 3i + pj − 2k and b = 4i − 2j + 5k perpendicular?

A	B	C	D
p = 1	p = −1	p = 2	p = 11

Q5 Calculate the scalar product of a = 2i − j + 3k and b = −3i + 4j + k.

A	B	C	D
−7	5	−3	−1

Q6 Find the cosine of the angle θ between a = 2i + j + 2k and b = i + 2j + 2k.

A	B	C	D
8/9	4/9	2/3	7/9

Q7 Which is a vector equation of the straight line through (2, −1, 5) with direction 3i + 4j − k?

A	B	C	D
r = (2i − j + 5k) + t(3i + 4j − k)	r = (3i + 4j − k) + t(2i − j + 5k)	r = (2i − j + 5k) + (3i + 4j − k)	r = (5i + 3j + 4k) + t(i − j + k)

Q8 Does the point P(5, 7, 1) lie on the line r = (1, 1, 3) + t(2, 3, −1)?

A	B	C	D
Yes, corresponding to t = 2	No, the z-component fails	Yes, corresponding to t = 3	No, the y-component fails

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Q9 Two lines have directions d1 = 2i − 6j + 4k and d2 = −i + 3j − 2k. The lines are:

A	B	C	D
Parallel, because d1 = −2d2	Perpendicular, because d1 · d2 = 0	Skew	Intersecting at 90°

Q10 Two non-parallel lines in 3D that do not intersect are called:

A	B	C	D
Skew lines	Asymptotic lines	Orthogonal lines	Coincident lines

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Q11 Find the acute angle between lines with direction vectors d1 = (1, 2, 2) and d2 = (2, −1, 2).

A	B	C	D
63.6°	26.4°	45.0°	90.0°

Q12 The foot of perpendicular from Q to a line r = a + tb occurs at point P where:

A	B	C	D				
QP · b = 0	QP · a = 0		QP	=	b		QP = b

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p23-9-vectors/