P23·7

Further algebra

CHAPTER ASSIGNMENT · 72 MARKS

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. Give exact simplified coefficients in binomial expansions and state validity ranges clearly using modulus or inequality notation.

Practice questions from *Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook* (Cambridge University Press, Sue Pemberton and Julianne Hughes), Chapter 7 (Further algebra), Exercises 7A to 7E and End-of-chapter review exercise 7, for classroom use. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 7A · Improper algebraic fractions

7A.1 Express each of the following improper fractions as the sum of a polynomial and a proper fraction: (a) $8x/(2x - 5)$ · (b) $(6x + 1)/(3x + 2)$ · (c) $(4x^3 - 3)/(2x + 1)$ · (d) $(x^3 + 4x^2 + 3x - 1)/(x^2 + 2x + 3)$ · (e) $(7x^3 + 2x^2 - 5x + 1)/(x^2 - 5)$ · (f) $(x^4 + 2x^2 - 5)/(x^2 + 1)$

7A.2 Given that $(x^3 + x^2 - 7)/(x - 3) \equiv Ax^2 + Bx + C + D/(x - 3)$, find the values of A, B, C and D.

7A.3 Given that $(x^4 + 5x^2 - 1)/(x + 1) \equiv Ax^3 + Bx^2 + Cx + D + E/(x + 1)$, find the values of A, B, C, D and E.

7A.4 Given that $(2x^4 + 3x^3 + 4x^2 + 5x + 6)/(x^3 + 2x) \equiv Ax + B + (Cx + D)/(x^3 + 2x)$, find the values of A, B, C and D.

Exercise 7B · Partial fractions

7B.1 Express the following proper fractions as partial fractions: (a) $(6x - 2)/((x - 2)(x + 3))$ · (b) $(7x + 12)/(2x(x - 4))$ · (c) $(15x + 13)/((x - 1)(3x + 1))$ · (d) $(x - 1)/((3x - 5)(x - 3))$ · (e) $(6x^2 + 5x - 2)/(x(x - 1)(2x + 1))$ · (f) $(11x + 12)/((2x + 3)(x + 2)(x - 3))$

7B.2 Express the following proper fractions as partial fractions: (a) $2x/(x + 2)^2$ · (b) $(11x^2 + 14x + 5)/((2x + 1)(x + 1)^2)$ · (c) $(x^2 - 2)/(x(x - 1)^2)$ · (d) $(36x^2 + 2x - 4)/((2x - 3)(2x + 1)^2)$ · (e) $3/((x + 2)(x - 2)^2)$ · (f) $(3x + 4)/((x + 2)(x - 1)^2)$

7B.3 Express the following proper fractions as partial fractions: (a) $(2x^2 - 3x + 2)/(x(x^2 + 1))$ · (b) $(3x^2 + 4x + 17)/((2x + 1)(x^2 + 5))$ · (c) $(2x^2 - 6x - 9)/((3x + 5)(2x^2 + 1))$ · (d) $(-6x^2 - 21x + 50)/((3x - 5)(2x^2 + 5))$

7B.4 Express the following improper fractions as partial fractions: (a) $(2x^2 + 3x + 4)/((x - 1)(x + 2))$ · (b) $(x^2 + 3)/(x^2 - 4)$ · (c) $(22 - 17x + 21x^2 - 4x^3)/((x - 4)(x^2 + 1))$ · (d) $(4x^3 + x^2 - 16x + 7)/(2x^3 - 4x^2 + 2x)$

7B.5 Find the values of A, B, C and D such that $(4x^3 - 9x^2 + 11x - 4)/(x^2(2x - 1)) \equiv A + B/x + C/x^2 + D/(2x - 1)$.

7B.6 (a) Factorise $2x^3 - 3x^2 - 3x + 2$ completely. (b) Hence express $(x^2 - 13x - 5)/(2x^3 - 3x^2 - 3x + 2)$ in partial fractions.

7B.7 (a) Factorise $2x^3 - 11x^2 + 12x + 9$ completely. (b) Hence express $(24 - x)/(2x^3 - 11x^2 + 12x + 9)$ in partial fractions.

7B.8 The polynomial $p(x) = 2x^3 + 5x^2 + ax + b$ is exactly divisible by $2x + 1$ and leaves a remainder of 9 when divided by $x + 2$. (a) Find the value of a and the value of b. (b) Factorise $p(x)$ completely. (c) Express $120/p(x)$ in partial fractions.

7B.9 (a) Express $2/(x(x + 2))$ in partial fractions. (b) Find an expression for the sum of the first n terms of the series: $2/(1 \times 3) + 2/(2 \times 4) + 2/(3 \times 5) + 2/(4 \times 6) + 2/(5 \times 7) + \ldots$ (c) Find the sum to infinity of this series.

7B.10 Investigate whether the series $1/(1 \times 2 \times 3) + 1/(2 \times 3 \times 4) + 1/(3 \times 4 \times 5) + 1/(4 \times 5 \times 6) + \ldots$ is a telescoping series. If it is a telescoping series, find an expression for the sum of the first n terms and the sum to infinity.

Exercise 7C · Binomial expansion of $(1 + x)^n$ for rational n

7C.1 Expand the following, in ascending powers of x, up to and including the term in x^3. State the range of values of x for which each expansion is valid: (a) $(1 + x)^{-2}$ · (b) $(1 + 3x)^{-1}$ · (c) $(1 - 2x)^{-4}$ · (d) $(1 + x/2)^{-3}$ · (e) $\sqrt{(1 + 2x)}$ · (f) $\sqrt[3]{(1 - 3x)}$ · (g) $2/(1 - 4x)^2$ · (h) $(1 - 3x)/(1 + 2x)^3$ · (i) $(1 + x)/\sqrt{(1 - 2x)}$

7C.2 Find the first 3 terms, in ascending powers of x, in the expansion of each of the following expressions. State the range of values of x for which each expansion is valid: (a) $(1 + x^2)^{-3}$ · (b) $\sqrt[3]{(1 - 2x^2)}$ · (c) $(\sqrt{(1 - 4x^2)})^3$

7C.3 Find the first 5 terms, in ascending powers of x, in the expansion of $(2 + 3x)/\sqrt{(1 - 5x^2)}$.

7C.4 Can the binomial expansion be used to expand each of these expressions? (a) $(3x - 1)^{-2}$ · (b) $\sqrt{(2x - 1)}$. Give reasons for your answers.

7C.5 Expand $(2x - 1)^{-3}$ in ascending powers of x, up to and including the term in x^3.

7C.6 Show that the expansion of $(1 + x)^{1/2}$ is not valid when $x = 3$.

7C.7 Given that $(1 - 3x)^{-4} - (1 + 2x)^{3/2} \approx 9x + kx^2$ for small values of x, find the value of the constant k.

7C.8 Given that $a/(1 - x) + b/(1 + 2x) \approx -3 + 12x$ for small values of x, find the value of a and the value of b.

7C.9 When $(1 + ax)^{-3}$, where a is a positive constant, is expanded the coefficients of x and x^2 are equal. (a) Find the value of a. (b) When a has this value, obtain the first 5 terms in the expansion.

7C.10 When $(3 - 2x)(1 + ax)^{2/3}$ is expanded the coefficient of x^2 is −15. Find the two possible values of a.

7C.11 The first 3 terms in the expansion of $(1 + ax)^n$ are $1 - 24x + 384x^2$. (a) Find the value of a and the value of n. (b) Hence find the term in x^3.

Exercise 7D · Binomial expansion of $(a + x)^n$ for rational n

7D.1 Expand the following, in ascending powers of x, up to and including the term in x^3. State the range of values of x for which each expansion is valid: (a) $(2 + x)^{-2}$ · (b) $(5 - 2x)^{-1}$ · (c) $\sqrt{(9 - x)}$ · (d) $\sqrt[3]{(3x + 8)}$ · (e) $4/(3 - x)^3$ · (f) $(1 + 2x)/(2x - 5)^3$

7D.2 Find the first 3 terms, in ascending powers of x, in the expansion of each of the following expressions. State the range of values of x for which each expansion is valid: (a) $(2 + x^2)^{-2}$ · (b) $\sqrt[3]{(8 - 3x^2)}$ · (c) $(\sqrt{(3 - x^2)})^5$

7D.3 Expand $(1 - 2x)\sqrt[3]{(8 - 2x)}$ in ascending powers of x, up to and including the term in x^3.

7D.4 (a) Expand $(2 - x)^{-1}$ and $(1 + 3x)^{-2}$ giving the first 3 terms in each expansion. (b) Use your answers to part a to find the first 3 terms in the expansion of $(2 - x)^{-1}(1 + 3x)^{-2}$, stating the range of values of x for which this expansion is valid.

7D.5 The first 3 terms in the expansion of $(a - 5x)^{-2}$ are $1/4 + (5/4)x + bx^2$. Find the value of a, the value of b and the term in x^3.

7D.6 In the expansion of $\sqrt{(3 + ax)}$ where $a \neq 0$, the coefficient of the term in x^2 is 3 times the coefficient of the term in x^3. Find the value of a.

7D.7 (a) Find the first 4 terms in the expansion of $(1 + 2/x)^{-1}$, where $|2/x| < 1$. (b) Show that $(1 + 2/x)^{-1} \equiv x/(x + 2) \equiv (x/2)(1 + x/2)^{-1}$. (c) Find the first 4 terms in the expansion of $(x/2)(1 + x/2)^{-1}$, where $|x/2| < 1$. (d) Explain why your expansions are different.

Exercise 7E · Partial fractions and binomial expansions

7E.1 (a) Express $(7x - 1)/((1 - x)(1 + 2x))$ in partial fractions. (b) Hence obtain the expansion of $(7x - 1)/((1 - x)(1 + 2x))$ in ascending powers of x, up to and including the term in x^3.

7E.2 (a) Express $(5x^2 + x)/((1 - x)^2(1 - 3x))$ in partial fractions. (b) Hence obtain the expansion of $(5x^2 + x)/((1 - x)^2(1 - 3x))$ in ascending powers of x, up to and including the term in x^3.

7E.3 (a) Express $(7x^2 + 4x + 4)/((1 - x)(2x^2 + 1))$ in partial fractions. (b) Hence obtain the expansion of $(7x^2 + 4x + 4)/((1 - x)(2x^2 + 1))$ in ascending powers of x, up to and including the term in x^3.

7E.4 (a) Express $(19 - 7x - 6x^2)/((2x + 1)(2 - 3x))$ in partial fractions. (b) Hence obtain the coefficient of x^2 in the expansion of $(19 - 7x - 6x^2)/((2x + 1)(2 - 3x))$.

7E.5 (a) Express $21/((x - 4)(x + 3))$ in partial fractions. (b) Hence obtain the expansion of $24/((x - 4)(x + 3))$ in ascending powers of x, up to and including the term in x^2.

7E.6 (a) Express $(6x^2 - 24x + 15)/((x + 1)(x - 2)^2)$ in partial fractions. (b) Hence obtain the expansion of $(6x^2 - 24x + 15)/((x + 1)(x - 2)^2)$ in ascending powers of x, up to and including the term in x^2.

End-of-chapter review exercise 7

R7.1 Expand $(1 - 2x)^{-4}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

R7.2 Expand $\sqrt[3]{(1 - 6x)}$ in ascending powers of x up to and including the term in x^3, simplifying the coefficients. [4]

R7.3 Expand $(2 - x)(1 + 2x)^{-3/2}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

R7.4 Expand $(1 + 3x)/\sqrt{(1 + 2x)}$ in ascending powers of x up to and including the term in x^2, simplifying the coefficients. [4]

R7.5 Expand $1/\sqrt{(1 + 3x)}$ in ascending powers of x, up to and including the term in x^3, simplifying the coefficients. [4]

R7.6 Expand $10/(2 - x)^2$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

R7.7 Expand $1/\sqrt{(4 - 5x)}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

R7.8 Express $12/(x^2(2x - 3))$ in the form $A/x + B/x^2 + C/(2x - 3)$. [4]

R7.9 Express $(8x^2 + 4x + 21)/((x + 2)(x^2 + 5))$ in the form $A/(x + 2) + (Bx + C)/(x^2 + 5)$. [4]

R7.10 Express $(7x^2 - 3x + 2)/(x(x^2 + 1))$ in partial fractions. [5]

R7.11 Given that $(9x^3 - 11x^2 + 8x - 4)/(x^2(3x - 2)) \equiv A + B/x + C/x^2 + D/(3x - 2)$, find the values of A, B, C and D. [5]

R7.12 Express $(4x^2 - 5x + 3)/((x + 2)(2x - 1))$ in partial fractions. [5]

R7.13 Show that for small values of x^2, $(1 - 2x^2)^{-2} - (1 + 6x^2)^{2/3} \approx kx^4$, where the value of the constant k is to be determined. [6]

R7.14 Let $f(x) = (4x^2 + 12)/((x + 1)(x - 3)^2)$. (i) Express f(x) in partial fractions. [5] (ii) Hence obtain the expansion of f(x) in ascending powers of x, up to and including the term in x^2. [5]

R7.15 Let $f(x) = (2x^2 - 7x - 1)/((x - 2)(x^2 + 3))$. (i) Express $f(x)$ in partial fractions. [5] (ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

R7.16 Let $f(x) = 3x/((1 + x)(1 + 2x^2))$. (i) Express $f(x)$ in partial fractions. [5] (ii) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^3. [5]

Check yourself

CY.1 Express $(2x^3 - x^2 + 4x - 7)/(x^2 - 1)$ as the sum of a linear polynomial and partial fractions.

CY.2 Decompose $(5x^2 - 4x + 7)/((x - 1)(x^2 + 2))$ into partial fractions.

CY.3 Decompose $(4x^2 + 5x - 3)/((x + 2)(x - 1)^2)$ into partial fractions.

CY.4 Expand $(4 - 2x)^{-1/2}$ in ascending powers of x up to and including the term in x^3, and state the set of values of x for which the expansion is valid.

CY.5 Let $f(x) = (8 - x)/((1 + 2x)(2 - x))$. Express f(x) in partial fractions and hence find the first four terms in its Maclaurin/binomial series expansion in ascending powers of x, stating the validity range.

Chapter quiz

Q1 Write the improper fraction $(3x + 5)/(x - 1)$ as a polynomial plus a proper fraction.

A	B	C	D
$3 + 8/(x - 1)$	$3 + 2/(x - 1)$	$3 - 8/(x - 1)$	$1 + 8/(x - 1)$

Q2 Given $(5x + 1)/((x - 1)(x + 2)) \equiv A/(x - 1) + B/(x + 2)$, find A and B.

A	B	C	D
$A = 3, B = 2$	$A = 2, B = 3$	$A = -2, B = 7$	$A = 4, B = 1$

Q3 Which partial fraction form is correct for $(3x + 7)/((x + 1)(x - 2)^2)$?

A	B	C	D
$A/(x + 1) + B/(x - 2)^2$	$A/(x + 1) + (Bx + C)/(x - 2)^2$	$A/(x + 1) + B/(x - 2) + C/(x - 2)^2$	$(Ax + B)/(x + 1) + C/(x - 2)^2$

Q4 Express $(2x - 5)/(x - 3)^2$ in partial fractions.

A	B	C	D
$2/(x - 3) - 5/(x - 3)^2$	$2/(x - 3) + 1/(x - 3)^2$	$1/(x - 3) + 2/(x - 3)^2$	$2/(x - 3) - 1/(x - 3)^2$

Q5 Which partial fraction form is used for $(4x^2 + 5)/((2x - 1)(x^2 + 3))$?

A	B	C	D
$A/(2x - 1) + B/(x^2 + 3)$	$A/(2x - 1) + B/(x + \sqrt{3}) + C/(x - \sqrt{3})$	$A/(2x - 1) + (Bx + C)/(x^2 + 3)$	$(Ax + B)/(2x - 1) + C/(x^2 + 3)$

Q6 Given $(7x^2 + 4x + 4)/((1 - x)(2x^2 + 1)) \equiv A/(1 - x) + (Bx + C)/(2x^2 + 1)$, find A.

A	B	C	D
$A = 3$	$A = 5$	$A = 7$	$A = -5$

Q7 Find the first three terms in the expansion of $(1 + x)^{-2}$ in ascending powers of x.

A	B	C	D
$1 - 2x + 3x^2$	$1 - 2x + 4x^2$	$1 + 2x + 3x^2$	$1 - 2x - 3x^2$

Q8 For which values of x is the binomial expansion of $(1 - 4x)^{-1/2}$ valid?

A	B	C	D
$\lvert x \rvert < 4$	$\lvert x \rvert < 1$	$\lvert x \rvert < 1/4$	$x < 1/4$

Q9 What is the coefficient of x^2 in the binomial expansion of $\sqrt{(1 + 6x)}$?

A	B	C	D
−9/2	9/2	−9/4	−3/8

Q10 How do you rewrite $(4 + x)^{-1/2}$ before applying the $(1 + u)^n$ binomial formula?

A	B	C	D
$4(1 + x/4)^{-1/2}$	$(1/2)(1 + x/4)^{-1/2}$	$(1/4)(1 + x/4)^{-1/2}$	$2(1 + 4x)^{-1/2}$

Q11 Find the first two terms in the expansion of $(8 - 3x)^{1/3}$ and its validity range.

A	B	C	D
$2 - x/4$, valid for $\lvert x \rvert < 8/3$	$2 - x$, valid for $\lvert x \rvert < 8/3$	$2 - x/4$, valid for $\lvert x \rvert < 3/8$	$8 - x$, valid for $\lvert x \rvert < 8$

Q12 If $f(x) = 2/(1 - x) - 1/(2 + x)$, for which x is the full binomial expansion valid?

A	B	C	D
$\lvert x \rvert < 2$	$\lvert x \rvert < 1$	$\lvert x \rvert < 1/2$	$-1 < x < 2$

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p23-7-further-algebra/