P23·6

Numerical solutions of equations

CHAPTER ASSIGNMENT · 362 MARKS

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. Give exact answers unless a question asks for a specified number of decimal places or significant figures. Work in radians for trigonometric functions unless degrees are stated.

Practice questions from *Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook* (Cambridge University Press, Sue Pemberton and Julianne Hughes), Chapter 6 (Numerical solutions of equations) and Cross-topic review exercise 2, for classroom use. Where a question uses a diagram, the diagram is described in words. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Prerequisite knowledge · Check your skills

PK.1 Evaluate each of the following expressions when $x = 1.5$.

(a) $(2/3)(x + 1/x^2)$ · (b) $100 \sin x - 99x$, (x in radians) · (c) e^{-x} · (d) $\ln x + 5$

PK.2 Rearrange each formula so that x is the subject.

(a) $y = 4x^2 + 13$, given $x > 0$ · (b) $0 = (y - 3x + 7)/w$ · (c) $3 = \sqrt[3]{((1 + y)/x^5)}$

PK.3 Show that the x-coordinates of the solutions to the simultaneous equations $y = x^2 - 6x + 10$ and $y = x$ also satisfy the equation $x^2 - 7x + 10 = 0$.

PK.4 Find the roots of the equation $x^2 - 7x + 10 = 0$.

PK.5 Sketch the graph, $y = f(x)$, of each function.

(a) $f(x) = x^3 + 1$ · (b) $f(x) = \cos 2x - 3$ · (c) $f(x) = 4x^2 - 16x$

PK.6 Sketch the graph, $y = g(x)$, of each function.

(a) $g(x) = e^x/2$ · (b) $g(x) = \ln(x - 1)$ · (c) $g(x) = \operatorname{cosec} x$

Exercise 6A · Finding a starting point

6A.1 (a) On the same axes, sketch the graphs of $y = \sqrt{(1 + x)}$ and $y = x^2$.

(b) Using your answer to part a, explain why the equation $x^2 - \sqrt{(1 + x)} = 0$ has two roots.

(c) Show, by calculation, that the smaller of these two roots lies between $x = -1$ and $x = 0$.

6A.2 (a) By sketching graphs of $y = x^3 + 5x^2$ and $y = 5 - 2x$, determine the number of real roots of the equation $x^3 + 5x^2 + 2x - 5 = 0$.

(b) Verify by calculation that the largest root of $x^3 + 5x^2 + 2x - 5 = 0$ lies between $x = 0$ and $x = 2$.

6A.3 The curve $y = x^3 + 5x - 1$ cuts the x-axis at the point $(\alpha, 0)$.

(a) By sketching the graph of $y = x^3$ and one other suitable graph, deduce that this is the only point where the curve $y = x^3 + 5x - 1$ cuts the x-axis.

(b) Using calculations, show that $0.1 < \alpha < 0.5$.

6A.4 (a) On the same axes, sketch the graphs of $y = \ln(x + 1)$ and $y = 3x - 4$.

(b) Hence deduce the number of roots of the equation $\ln(x + 1) - 3x + 4 = 0$.

6A.5 (a) By sketching graphs of $y = e^x$ and $y = x + 6$, determine the number of roots of the equation $e^x - x - 6 = 0$.

(b) Show, by calculation, that $e^x - x - 6 = 0$ has a root that lies between $x = 2.0$ and $x = 2.1$.

6A.6 (a) Show, by calculation, that $(x + 2)e^{5x} = 1$ has a root between $x = 0$ and $x = -0.2$.

(b) Show, by sketching the graph of $y = e^{5x}$ and one other suitable graph, that this is the only root of this equation.

6A.7 (a) Show by calculation that the equation $\cos^{-1}2x = 1 - x$ has a root α, where $0.4 \leq \alpha \leq 0.5$.

(b) A diagram shows the graph of $y = \cos^{-1}2x$ where $-0.5 \leq x \leq 0.5$, which is a continuous decreasing curve passing through $(-0.5, \pi)$, $(0, \pi/2)$ and $(0.5, 0)$. By drawing a suitable graph on a sketch of this diagram, show that α is the only root of $\cos^{-1}2x = 1 - x$ for $-0.5 \leq x \leq 0.5$.

6A.8 The curve with equation $y = (\sin x)/(2x + 3)$ where $x < -\pi/2$ intersects the line $y = 1$ at the point $(\alpha, 1)$.

(a) By sketching the graph of $y = \sin x$ for $-2\pi < x < -\pi/2$ radians and one other suitable graph, deduce that this is the only point where the curve $y = (\sin x)/(2x + 3)$ intersects the line $y = 1$.

(b) Using calculations, show that $-2 < \alpha < -1.9$.

6A.9 (a) By sketching a suitable pair of graphs, show that the equation $x^3 + 4x = 7x + 4$ has only one root for $0 \leq x \leq 5$.

(b) Verify by calculation that this root lies between $x = 2$ and $x = 3$.

6A.10 (a) Show graphically that the equation $2^x = x + 4$ has exactly two roots.

(b) Show, by calculation, that the larger of the two roots is between 2.7 and 2.8.

6A.11 (a) By sketching two suitable graphs on the same diagram, show that the equation $\cot x = x^2$ has one root between 0 and $\pi/2$ radians.

(b) Show, by calculation, that this root, α, lies between 0.8 and 1.

6A.12 (a) By sketching a suitable pair of graphs, deduce the number of roots of the equation $x = \tan 2x$ for $0 < x < 2\pi$.

(b) Verify, by calculation, that one of these roots, α, lies between 2.1 and 2.2.

6A.13 (a) Show graphically that the equation $\operatorname{cosec} x = \sin x$ has exactly two roots for $0 < x < 2\pi$.

(b) Use an algebraic method to find the value of the larger root correct to 3 significant figures.

6A.14 $f(x) = 20x^3 + 8x^2 - 7x - 3$.

(a) The equation $20x^3 + 8x^2 - 7x - 3 = 0$ has exactly two roots. Without factorising the cubic equation, show, by calculation, that one of these roots is between 0.5 and 1.

(b) Show, by sketching the graph of $y = f(x)$ for $-1 \leq x \leq 1$, that the smaller root is between -1 and 0.

(c) Explain why it is not necessary to use a numerical method to find the two solutions of this equation.

6A.15 A guitar tuning peg is in the shape of a cylinder with a hemisphere at one end. The cylinder is 20 mm long and the whole peg is made from 800 mm^3 of plastic. The base radius of the cylinder is r mm.

(a) Show that $\pi r^3 + 30\pi r^2 - 1200 = 0$.

(b) Show that the value of r is between 3 mm and 4 mm.

Explore 6.1 · When the sign-change test needs care

E6.1.1 Anil is trying to find the roots of $f(x) = |\tan x| = 0$ between 0 and 2π radians. He states: "f(3.1) > 0 and f(3.2) > 0. There is no change of sign so there is no root of $|\tan x| = 0$ between 3.1 and 3.2." Mika states: "$y = |\tan x|$ and it meets the x-axis where $x = \pi$ radians so Anil has made a calculation error." Discuss Anil and Mika's statements.

Exercise 6B · Improving your solution using iteration

6B.1 The equation $x^3 + 5x - 7 = 0$ has a root α between 1 and 1.2. The iterative formula $x_{n+1} = (7 - x_n^3)/5$ can be used to find the value of α.

(a) Using a starting value of $x_1 = 1.1$, find and write down, correct to 4 decimal places, the value of each of x_2, x_3, x_4, x_5, x_6.

(b) Write the value of x_6 correct to 2 decimal places and prove that this is the value of α correct to 2 decimal places.

6B.2 (a) Show that the equation $\ln(x + 1) + 2x - 4 = 0$ has a root α between $x = 1$ and $x = 2$.

(b) Use the iterative formula $x_{n+1} = (4 - \ln(x_n + 1))/2$ with an initial value of $x_1 = 1.5$ to find the value of α correct to 3 decimal places.

6B.3 The equation $(x - 0.5)e^{3x} = 1$ has a root α.

(a) Show, by sketching the graph of $y = e^{3x}$ and one other suitable graph, that α is the only root of this equation.

(b) Use the iterative formula $x_{n+1} = e^{-3x_n} + 0.5$ with $x_1 = 0.5$ to find the value of α correct to 2 decimal places. Give the value of each of your iterations to 4 decimal places.

6B.4 (a) By sketching a suitable pair of graphs, show that the equation $x^3 + 10x = x + 5$ has only one root that lies between 0 and 1.

(b) Use the iterative formula $x_{n+1} = (5 - x_n^3)/9$, with a suitable value for x_1, to find the value of this root correct to 4 decimal places. Give the result of each iteration to 6 decimal places.

6B.5 The equation $\cos^{-1}3x = 1 - x$ has a root α.

(a) Show, by calculation, that α is between $\pi/15$ and $\pi/12$.

(b) Show that the given equation can be rearranged into the form $x = (1/3)\cos(1 - x)$.

(c) Using the iterative formula $x_{n+1} = (1/3)\cos(1 - x_n)$ with a suitable starting value, x_1, find the value of α correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

6B.6 The terms of a sequence with first term $x_1 = 1$ are defined by the iterative formula: $x_{n+1} = \sqrt{((5 - 2x_n - x_n^3)/5)}$. The terms converge to the value α.

(a) Use this formula to find the value of α correct to 2 decimal places. Give the value of each term you calculate to 4 decimal places.

(b) The value α is the root of an equation of the form $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are integers. Find this equation.

6B.7 The equation $x^4 - 1 - x = 0$ has a root, α, between $x = 1$ and $x = 2$.

(a) Show that α also satisfies the equation $x = (1 + x)^{1/4}$.

(b) Write down an iterative formula based on the equation in part a.

(c) Use your iterative formula, with a starting value of $x_1 = 1.5$, to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

6B.8 The equation $\text{cosec } x = x^2$ has a root, α, between 1 and 2. The equation can be rearranged either as $x = \sin^{-1}(1/x^2)$ or $x = \sqrt{(1/\sin x)}$.

(a) Write down two possible iterative formulae, one based on each given rearrangement. Use the starting value 1.5.

(b) Show that one of the formulae from part a fails to converge.

(c) Show that the other formula from part a converges to α and find the value of α correct to 3 decimal places. Give the result of each iteration to 5 decimal places.

6B.9 The terms of a sequence, defined by the iterative formula $x_{n+1} = \ln(x_n^2 + 4)$, converge to the value α. The first term of the sequence is 2.

(a) Find the value of α correct to 2 decimal places. Give each term of the sequence you find to 4 decimal places.

(b) The value α is a root of an equation of the form $x^2 = f(x)$. Find this equation.

6B.10 The equation $(\sin(x - 1))/(2x - 3) + 1 = 0$ has a root, α, between x = 1 and x = 1.4.

(a) Show that α also satisfies the equation $x = (3 - \sin(x - 1))/2$.

(b) Using an iterative formula based on the equation from part a, with a suitable starting value, find the value of α correct to 3 significant figures. Give the result of each iteration to 5 significant figures.

6B.11 The graphs of $y = (3/2)^x - 1$ and y = x intersect at the points O(0, 0) and A.

(a) Sketch these graphs on the same diagram.

(b) Using logarithms, find a suitable iterative formula that can be used to find the coordinates of the point A.

(c) Calculate the length of the line OA, giving your answer correct to 2 significant figures. Give the value of any iterations you calculate to a suitable number of significant figures.

6B.12 A container is in the shape of a cone with a cylinder on top. The total height of the container is 33 cm, and the height of the cylinder is 3 times its base radius, r. The volume of the container must be 5500 cm^3. The base of the cone has a radius of r cm.

(a) Write down an expression for the height of the cone in terms of r.

(b) Show that $2\pi r^3 + 11\pi r^2 - 5500 = 0$.

(c) The equation in part b has a root, α. Show that α is also a root of the equation $r = \sqrt[3]{((5500 - 11\pi r^2)/(2\pi))}$.

(d) Using an iterative formula based on the equation in part c, and a starting value of 8, find the value of α correct to 3 decimal places. Give the value of each of your iterations to an appropriate degree of accuracy.

(e) Explain what information is given by your answer to part d, in the context of the question.

6B.13 The equation $x^3 - 7x^2 + 1 = 0$ has two positive roots, α and β, which are such that α lies between 0 and 1 and β lies between 6 and 7. By deriving two suitable iterative formulae from the given equation, carry out suitable iterations to find the value of α and of β, giving each correct to 2 decimal places. Give the value of each of your iterations to 4 decimal places.

6B.14 (a) Show graphically that the equation $\cot x = \sin x$ has a root, α, which is such that $0 < \alpha < \pi/2$.

(b) Show that the equation in part a can be rearranged as $x = \sin^{-1}(\sqrt{(\cos x)})$.

(c) Using an iterative formula based on the equation in part b, with an initial value of 0.9, find the value of α correct to 2 decimal places. Give the value of each iteration to an appropriate number of decimal places.

(d) The equation given in part a can be rearranged in other ways. Find an example of a formula, based on a different rearrangement of the equation given in part a, that also converges to the value of α, with an initial value of 0.9, in fewer iterations than the formula used in part c.

Explore 6.2 · Why some rearrangements fail to converge

E6.2.1 Consider why rearrangements of equations into the form $x = F(x)$ sometimes fail to converge.

(a) Discuss why formulae based on $x = (1 - x)^{1/5}$ (cobweb pattern) and $x = \ln(2x + 1)$ (staircase pattern) each converged to a root.

(b) Look at the equation $e^x = 3x + 1$ and the rearrangement $x = (e^x - 1)/3$. Does this give a convergent iterative formula for the positive root? Explain your answer using the graphs of $y = x$ and $y = (e^x - 1)/3$.

Exercise 6C · Using iterative processes in other areas of mathematics

6C.1 The curve with equation $y = (e^{2x} + x)/x^3$ has a stationary point with x-coordinate lying between 1 and 2.

(a) Show that the x-coordinate of this stationary point satisfies the equation $x = 3/2 + x/e^{2x}$.

(b) Using a suitable iterative process based on the equation given in part a, find the value of the x-coordinate of this stationary point correct to 4 decimal places. Give each iteration to 6 decimal places.

6C.2 The parametric equations of a curve are $x = t^2 + 6$, $y = t^4 - t^3 - 5t$. The curve has a stationary point for a value of t that lies between 1 and 2.

(a) Show that the value of t at this stationary point satisfies the equation $t = \sqrt[3]{((3t^2 + 5)/4)}$.

(b) Use an iterative process based on the equation in part a to find the value of t correct to 3 decimal places. Show the result of each iteration to 6 decimal places.

(c) Hence find the coordinates of the stationary point, giving each coordinate correct to 1 significant figure.

6C.3 In a diagram, triangle ABC is right-angled at B and angle BAC is θ radians. The point O is the midpoint of AC and OC = r. Angle BOC is 2θ radians and BOC is a sector of the circle with centre O. The area of triangle ABC is 2 times the area of the shaded segment between arc BC and chord BC.

(a) Show that θ satisfies the equation $\sin 2\theta = \theta$.

(b) This equation has one root in the interval $0 < \theta < \pi/2$. Use the iterative formula $\theta_{n+1} = \sin 2\theta_n$ to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

6C.4 A diagram shows the curve $y = x^2 \cos 4x$ for $0 \leq x \leq \pi/8$. The point P is a maximum point.

(a) Show that the x-coordinate of P satisfies the equation $4x^2 \tan 4x = 2x$.

(b) Show also that the x-coordinate of P satisfies the equation $x = (1/4) \tan^{-1}(1/(2x))$.

(c) Using an iterative formula based on the equation in part b with initial value $x_1 = 0.3$, find the x-coordinate of P correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

(d) [P3] Use integration by parts twice to find the exact area enclosed between the curve and the x-axis from 0 to $\pi/8$.

Explore 6.3 · The golden ratio

E6.3.1 When a rectangle with sides in the ratio 1 : φ (phi) can be cut into a square of side 1 and a smaller rectangle, also with sides in the ratio 1 : φ, the original rectangle is called a golden rectangle and the ratio 1 : φ is called the golden ratio. This means that $(\varphi - 1)/1 = 1/\varphi$. By using a suitable iterative formula based on this ratio, with a starting value of 1, find the value of φ correct to 5 significant figures.

End-of-chapter review exercise 6

R6.1 The terms of the sequence generated by the iterative formula $x_{n+1} = (6/7)(x_n + 1/x_n^3)$ with initial value $x_1 = 1.5$, converge to α.

(a) Use this formula to find α correct to 2 decimal places. Give the result of each iteration to an appropriate number of decimal places. [3]

(b) State an equation satisfied by α, and hence find the exact value of α. [2]

R6.2 The equation $x^3 + x/2 - 8 = 0$ has one real root, denoted by α.

(a) Find, by calculation, the pair of consecutive integers between which α lies. [2]

(b) Show that, if a sequence of values given by the iterative formula $x_{n+1} = \sqrt{(8/x_n - 1/2)}$ converges, then it converges to α. [2]

(c) Use this iterative formula to determine α correct to 1 decimal place. Give the result of each iteration to 4 decimal places. [3]

R6.3 (a) By sketching a suitable pair of graphs, show that the equation $e^{2x+1} = 14 - x^3$ has exactly one real root. [3]

(b) Show by calculation that this root lies between 0.5 and 1. [2]

(c) Show that this root also satisfies the equation $x = (\ln(14 - x^3) - 1)/2$. [1]

(d) Use an iteration process based on the equation in part c, with a suitable starting value, to find the root correct to 4 decimal places. Give the result of each step of the process to 6 decimal places. [3]

R6.4 The sequence of values given by the iterative formula $x_{n+1} = (x_n(1 + \sec^2 x_n) - \tan x_n)/(\sec^2 x_n - 1)$ with initial value $x_1 = 1$, converges to α.

(a) Use this formula to find α correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

(b) Show that α also satisfies the equation $\tan x - 2x = 0$ where $1 \leq x < \pi/2$. [2]

R6.5 The equation $(\ln x^2 - 6)/2 = x - 5$ has a root α, such that $p < \alpha < q$, where p and q are consecutive integers.

(a) Show that α also satisfies the equation $x = \ln x + 2$. [1]

(b) Find the value of p and the value of q. [2]

(c) Use the iterative formula based on the equation in part a with starting value x = p, to find the value of α correct to 3 decimal places. [3]

(d) Hence find an approximate value of y that satisfies the equation $5^y - 2 = y \ln 5$. [2]

R6.6 (a) By sketching a suitable pair of graphs, show that the equation $7 - x^6 = |x^2 - 1|$ has exactly two real roots, α and β, where α is a positive constant. [3]

(b) Show that α satisfies the equation $x = (8/x - x)^{1/5}$. [1]

(c) Write down an iterative formula based on the equation in part b. Use this formula, with a suitable starting value, to find the value of α correct to 3 significant figures. Give the result of each iteration to 5 significant figures. [3]

(d) Hence write down the value of β correct to 3 significant figures. [1]

R6.7 It is given that $\int_0^a (1/(3x + 5) + 2e^{6x})\, dx = 0.6$, where the constant $a > 0$.

(a) Show that $a = (1/6) \ln(2.8 + \ln(5/(3a + 5)))$. [4]

(b) Use an iterative formula based on the equation in part a, with a starting value of $a = 0.2$, to find the value of a correct to 3 decimal places. Give the result of each iteration to an appropriate number of decimal places. [3]

R6.8 A curve has parametric equations $x = 1/t$, $y = \sqrt{(t + 1)}$. The point P on the curve has parameter p. It is given that the gradient of the curve at P is -0.4.

(a) Show that $p = (2\sqrt{5} / 5)(p + 1)^{1/4}$. [3]

(b) Use an iterative process based on the equation in part a to find the value of p correct to 3 decimal places. Use a starting value of 1 and show the result of each iteration to 5 decimal places. [3]

R6.9 A, B and C are points on the circumference of the circle with centre O and radius r, where OC bisects angle ACB. The shaded region, ABC, is a sector of a circle with centre C and radius CA = CB. Angle OCA is equal to θ radians. The area of the shaded region is equal to 3/8 of the area of the circle with centre O.

(a) Show that $\theta = \cos^{-1}(\sqrt{(3\pi/(32\theta))})$. [3]

(b) Verify by calculation that θ lies between 0.8 and 1.2 radians. [2]

(c) Use the iterative formula $\theta_{n+1} = \cos^{-1}(\sqrt{(3\pi/(32\theta_n))})$ to find the value of θ correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

R6.10 The functions f and g are defined, for $0 \leq x \leq \pi$, by $f(x) = e^{x-2}$ and $g(x) = 5 - \cos x$. The gradients of the curves $y = f(x)$ and $y = g(x)$ are equal both when $x = p$ and when $x = q$.

(a) Given that $p < q$, verify by calculation that p is 0.16 correct to 2 decimal places. [4]

(b) Show that q satisfies the equation $q = 2 + \ln(\sin q)$. [1]

(c) Given also that $1.5 < q < 2.5$, use the iterative formula $q_{n+1} = 2 + \ln(\sin q_n)$ to calculate q correct to 2 decimal places, showing the result of each iteration to 4 decimal places. [3]

R6.11 The equation $x = \cos x + \sin x$ has a root α that lies between 1 and 1.4.

(a) Show that α is also a root of the equation $x = \sqrt{2}\cos(x - \pi/4)$. [3]

(b) Using the iterative formula $x_{n+1} = \sqrt{2}\cos(x_n - \pi/4)$, find the value of α, giving your answer correct to 2 decimal places. [3]

R6.12 [P3] (a) By sketching each of the graphs of $y = \sec x$ and $y = (\pi/2 - x)(\pi/4 + x)$ for $-\pi/2 < x \leq \pi/2$ on the same diagram, show that the equation $\sec x = (\pi/2 - x)(\pi/4 + x)$ has exactly two real roots in the interval $-\pi/2 < x \leq \pi/2$. [3]

(b) Show that the equation $\sec x = (\pi/2 - x)(\pi/4 + x)$ can be written in the form $x = \sqrt{((2\pi x + \pi^2 - 8\sec x)/8)}$. [1]

(c) The two real roots of the equation $\sec x = (\pi/2 - x)(\pi/4 + x)$ in the interval $-\pi/2 < x \leq \pi/2$ are denoted α and β. Verify by calculation that the smaller root, α, is -0.21 correct to 2 decimal places. [2]

(d) Using an iterative formula based on the equation given in part b, with an initial value of 1, find the value of β correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

R6.13 [P3] A diagram shows part of the curve $y = \cos^3 x$, where x is in radians. The shaded region between the curve, the axes and the line $x = \alpha$ (where $0 < \alpha < \pi/3$) is denoted by R. The area of R is equal to 0.3.

(a) Using the substitution $u = \sin x$, find $\int_0^{\alpha} \cos^3 x \, dx$. Hence show that $\sin \alpha = 0.9/(3 - \sin^2 \alpha)$. [6]

(b) Use the iterative formula $\alpha_{n+1} = \sin^{-1}(0.9/(3 - \sin^2\alpha_n))$ with $\alpha_1 = 0.2$, to find the value of α correct to 3 significant figures. Give the result of each iteration to 5 significant figures. [3]

R6.14 [P3] It is given that $\int_1^a \ln x \, dx = 5$, where a is a constant greater than 1.

(a) Show that $a = (4 + a)/\ln a$. [5]

(b) Use an iterative formula based on the equation in part a to find the value of a correct to 3 decimal places. Use an initial value of 5 and give the result of each iteration to 5 decimal places. [3]

Cross-topic review exercise 2 (Chapters 1–6)

CT2.1 The equation of a curve is $y = (1 + x)/(1 + 2x)$ for $x > -1/2$. Show that the gradient of the curve is always negative. [3]

CT2.2 A diagram shows the part of the curve $y = (1/2)\tan 2x$ for $0 \leq x \leq (1/2)\pi$, with a vertical asymptote at $x = (1/4)\pi$. Find the x-coordinates of the points on this part of the curve at which the gradient is 4. [5]

CT2.3 The parametric equations of a curve are $x = \ln(1 - 2t)$, $y = 2/t$, for $t < 0$.

(i) Show that $dy/dx = (1 - 2t)/t^2$. [3]

(ii) Find the exact coordinates of the only point on the curve at which the gradient is 3. [3]

CT2.4 The equation of a curve is $x^2y + y^2 = 6x$.

(i) Show that $dy/dx = (6 - 2xy)/(x^2 + 2y)$. [4]

(ii) Find the equation of the tangent to the curve at the point with coordinates (1, 2), giving your answer in the form $ax + by + c = 0$. [3]

CT2.5 (i) [P2] Use the trapezium rule with two intervals to estimate the value of $\int_0^1 1/(6 + 2e^x)\,dx$, giving your answer correct to 2 decimal places. [3]

(ii) Find $\int (e^x - 2)^2/e^{2x}\,dx$. [4]

CT2.6 (i) By sketching a suitable pair of graphs, show that the equation $\ln x = 4 - (1/2)x$ has exactly one real root, α. [2]

(ii) Verify by calculation that $4.5 < \alpha < 5.0$. [2]

(iii) Use an iterative formula $x_{n+1} = 8 - 2 \ln x_n$ to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

CT2.7 The parametric equations of a curve are $x = 1/\cos^3 t$, $y = \tan^3 t$, where $0 \leq t < (1/2)\pi$.

(i) Show that $dy/dx = \sin t$. [4]

(ii) Hence show that the equation of the tangent to the curve at the point with parameter t is $y = x \sin t - \tan t$. [3]

CT2.8 It is given that the positive constant a is such that $\int_{-a}^{a} (4e^{2x} + 5)\, dx = 100$.

(i) Show that $a = (1/2) \ln(50 + e^{-2a} - 5a)$. [4]

(ii) Use the iterative formula $a_{n+1} = (1/2) \ln(50 + e^{-2a_n} - 5a_n)$ to find a correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

CT2.9 The equation of a curve is $y = e^{-2x} \tan x$, for $0 \leq x < (1/2)\pi$.

(i) Obtain an expression for dy/dx and show that it can be written in the form $e^{-2x}(a + b \tan x)^2$, where a and b are constants. [5]

(ii) Explain why the gradient of the curve is never negative. [1]

(iii) Find the value of x for which the gradient is least. [1]

CT2.10 (i) Prove that $\tan\theta + \cot\theta \equiv 2/\sin 2\theta$. [3]

(ii) Hence:

(a) find the exact value of $\tan((1/8)\pi) + \cot((1/8)\pi)$, [2]

(b) evaluate $\int_0^{(1/2)\pi} 6/(\tan\theta + \cot\theta)\, d\theta$. [3]

CT2.11 The equation of a curve is $xy(x - 6y) = 9a^3$, where a is a non-zero constant. Show that there is only one point on the curve at which the tangent is parallel to the x-axis, and find the coordinates of this point. [7]

CT2.12 The curve with equation $y = 6/x^2$ intersects the line $y = x + 1$ at the point P.

(i) Verify by calculation that the x-coordinate of P lies between 1.4 and 1.6. [2]

(ii) Show that the x-coordinate of P satisfies the equation $x = \sqrt{(6/(x + 1))}$. [2]

(iii) Use the iterative formula $x_{n+1} = \sqrt{(6/(x_n + 1))}$, with initial value $x_1 = 1.5$, to determine the x-coordinate of P correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

CT2.13 The parametric equations of a curve are $x = e^{2t}$, $y = 4te^t$.

(i) Show that $dy/dx = 2(t + 1)/e^t$. [4]

(ii) Find the equation of the normal to the curve at the point where $t = 0$. [4]

CT2.14 For each of the following curves, find the exact gradient at the point indicated:

(i) $y = 3 \cos 2x - 5 \sin x$ at $((1/6)\pi, -1)$, [3]

(ii) $x^3 + 6xy + y^3 = 21$ at (1, 2). [5]

CT2.15 (i) By differentiating 1/cos x, show that if y = sec x then dy/dx = sec x tan x. [2]

(ii) Show that $1/(\sec x - \tan x) \equiv \sec x + \tan x$. [1]

(iii) Deduce that $1/(\sec x - \tan x)^2 \equiv 2\sec^2 x - 1 + 2\sec x \tan x$. [2]

(iv) Hence show that $\int_0^{(1/4)\pi} 1/(\sec x - \tan x)^2 \, dx = (1/4)(8\sqrt{2} - \pi)$. [3]

CT2.16 A curve is defined by the parametric equations $x = 2\tan\theta$, $y = 3\sin 2\theta$, for $0 \leq \theta < (1/2)\pi$.

(i) Show that $dy/dx = 6\cos^4\theta - 3\cos^2\theta$. [4]

(ii) Find the coordinates of the stationary point. [3]

(iii) Find the gradient of the curve at the point $(2\sqrt{3}, (3/2)\sqrt{3})$. [2]

CT2.17 A diagram shows the curve $y = 4e^{(1/2)x} - 6x + 3$ and its minimum point M above the positive x-axis.

(i) Show that the x-coordinate of M can be written in the form ln a, where the value of a is to be stated. [5]

(ii) Find the exact value of the area of the region enclosed by the curve and the lines $x = 0$, $x = 2$ and $y = 0$. [4]

CT2.18 The equation of a curve is $x^3 - 3x^2y + y^3 = 3$.

(i) Show that $dy/dx = (x^2 - 2xy)/(x^2 - y^2)$. [4]

(ii) Find the coordinates of the points on the curve where the tangent is parallel to the x-axis. [5]

CT2.19 A curve has parametric equations $x = 1/(2t + 1)^2$, $y = \sqrt{(t + 2)}$. The point P on the curve has parameter p and it is given that the gradient of the curve at P is −1.

(i) Show that $p = (p + 2)^{1/6} - 1/2$. [6]

(ii) Use an iterative process based on the equation in part i to find the value of p correct to 3 decimal places. Use a starting value of 0.7 and show the result of each iteration to 5 decimal places. [3]

CT2.20 (i) Find:

(a) $\int (e^{2x} + 6)/e^{2x}\, dx$, [3]

(b) $\int 3\cos^2 x\, dx$. [3]

(ii) [P2] Use the trapezium rule with 2 intervals to estimate the value of $\int_1^2 6/\ln(x + 2)\, dx$, giving your answer correct to 2 decimal places. [3]

CT2.21 (i) By sketching a suitable pair of graphs, show that the equation $e^{2x} = 14 - x^2$ has exactly two real roots. [3]

(ii) Show by calculation that the positive root lies between 1.2 and 1.3. [2]

(iii) Show that this root also satisfies the equation $x = (1/2) \ln(14 - x^2)$. [1]

(iv) Use an iteration process based on the equation in part iii, with a suitable starting value, to find the root correct to 2 decimal places. Give the result of each step of the process to 4 decimal places. [3]

CT2.22 (i) By sketching each of the graphs $y = \text{cosec } x$ and $y = x(\pi - x)$ for $0 < x < \pi$, show that the equation $\text{cosec } x = x(\pi - x)$ has exactly two real roots in the interval $0 < x < \pi$. [3]

(ii) Show that the equation $\text{cosec } x = x(\pi - x)$ can be written in the form $x = (1 + x^2 \sin x)/(\pi \sin x)$. [2]

(iii) The two real roots of the equation $\text{cosec } x = x(\pi - x)$ in the interval $0 < x < \pi$ are denoted by α and β, where $\alpha < \beta$.

(a) Use the iterative formula $x_{n+1} = (1 + x_n^2 \sin x_n)/(\pi \sin x_n)$ to find α correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

(b) Deduce the value of β correct to 2 decimal places. [1]

CT2.23 A diagram shows part of the curve with parametric equations $x = 2 \ln(t + 2)$, $y = t^3 + 2t + 3$, passing through the origin O and a point P in the third quadrant.

(i) Find the gradient of the curve at the origin. [5]

(ii) At the point P on the curve, the value of the parameter is p. It is given that the gradient of the curve at P is 1/2.

(a) Show that $p = 1/(3p^2 + 2) - 2$. [1]

(b) By first using an iterative formula based on the equation in part a, determine the coordinates of the point P. Give the result of each iteration to 5 decimal places and each coordinate of P correct to 2 decimal places. [4]

CT2.24 (i) Prove that $2 \operatorname{cosec} 2\theta \tan \theta \equiv \sec^2\theta$. [3]

(ii) Hence:

(a) solve the equation $2 \operatorname{cosec} 2\theta \tan \theta = 5$ for $0 < \theta < \pi$, [3]

(b) find the exact value of $\int_0^{(1/6)\pi} 2 \operatorname{cosec} 4x \tan 2x \, dx$. [4]

CT2.25 (i) Prove the identity $\cos 4\theta + 4\cos 2\theta \equiv 8\cos^4\theta - 3$. [4]

(ii) Hence:

(a) solve the equation $\cos 4\theta + 4\cos 2\theta = 1$ for $-(1/2)\pi \leq \theta \leq (1/2)\pi$, [3]

(b) find the exact value of $\int_0^{(1/4)\pi} \cos^4\theta \, d\theta$. [3]

CT2.26 (i) By first expanding $\cos(2x + x)$, show that $\cos 3x = 4\cos^3 x - 3\cos x$. [5]

(ii) Hence show that $\int_0^{(1/6)\pi} (2\cos^3 x - \cos x) \, dx = 5/12$. [5]

CT2.27 A diagram shows the damped wave curve $y = 10e^{-(1/2)x} \sin 4x$ for $x \geq 0$. The stationary points are labelled T_1, T_2, T_3, ... in order of increasing x.

(i) Find the x-coordinates of T_1 and T_2, giving each x-coordinate correct to 3 decimal places. [6]

(ii) It is given that the x-coordinate of T_n is greater than 25. Find the least possible value of n. [4]

CT2.28 The equation of a curve is $y = 3x^2/(x^2 + 4)$. At the point on the curve with positive x-coordinate p, the gradient of the curve is 1/2.

(i) Show that $p = \sqrt{((48p - 16)/(p^2 + 8))}$. [5]

(ii) Show by calculation that $2 < p < 3$. [2]

(iii) Use an iterative formula based on the equation in part i to find the value of p correct to 4 significant figures. Give the result of each iteration to 6 significant figures. [3]

CT2.29 (i) By differentiating $1/\cos\theta$, show that if $y = \sec\theta$ then $dy/d\theta = \tan\theta \sec\theta$. [3]

(ii) Hence show that $d^2y/d\theta^2 = a\sec^3\theta + b\sec\theta$, giving the values of a and b. [4]

(iii) Find the exact value of $\int_0^{(1/4)\pi} (1 + \tan^2\theta - 3\sec\theta\tan\theta)\, d\theta$. [5]

Check yourself

CY.1 Show by calculation that the equation $x^3 + 3x - 9 = 0$ has a root α between 1 and 2, and explain why it is the only real root. [3]

CY.2 Using the iterative formula $x_{n+1} = \sqrt[3]{(9 - 3x_n)}$ with $x_1 = 1.5$, find the root α of $x^3 + 3x - 9 = 0$ correct to 3 decimal places, giving each iteration to 5 decimal places. [4]

CY.3 Prove by a change-of-sign test that the root α in Question CY.2 is 1.634 correct to 3 decimal places. [2]

CY.4 The sequence $x_{n+1} = \ln(4x_n + 3)$ with $x_1 = 2$ converges to a root α. State the equation in the form $e^x = ax + b$ satisfied by α, and find α correct to 2 decimal places. [4]

CY.5 For the equation $x^2 - 5x + 2 = 0$, compare the two rearrangements $x = (x^2 + 2)/5$ and $x = \sqrt{(5x - 2)}$ near the larger root $\beta \approx 4.56$ by evaluating $|F'(\beta)|$ for each, and state which iterative formula converges to β. [4]

CY.6 The curve $y = x\,e^{-x} + 2x - 1$ crosses the x-axis at $x = \alpha$. Show that $0.3 < \alpha < 0.4$, and use $x_{n+1} = (1 - x_n\,e^{-x_n})/2$ with $x_1 = 0.35$ to find α correct to 3 decimal places. [5]

Chapter quiz

Q1 For $f(x) = x^3 + 2x - 5$, between which consecutive integers does the real root α lie?

A	B	C	D
Between 1 and 2	Between 0 and 1	Between 2 and 3	Between −1 and 0

Q2 Which pair of graphs can be sketched to find the number of real roots of $e^x - 3x - 2 = 0$?

A	B	C	D
$y = e^x$ and $y = 3x + 2$	$y = e^x$ and $y = 3x - 2$	$y = e^x$ and $y = -3x - 2$	$y = \ln x$ and $y = 3x + 2$

Q3 When can a sign-change check on [a, b] fail to detect a root of a continuous function $f(x) = 0$?

A	B	C	D
When $y = f(x)$ touches the x-axis or crosses it an even number of times	When $f(x)$ is strictly increasing on [a, b]	When a and b have the same sign	When the gradient $f'(x)$ is negative on [a, b]

Q4 Which rearrangement of $x^3 - 4x - 1 = 0$ into the form $x = F(x)$ is algebraically valid?

A	B	C	D
$x = \sqrt[3]{(4x + 1)}$	$x = \sqrt[3]{(4x - 1)}$	$x = (x^3 + 1)/4$	$x = \sqrt{(4 + 1/x^2)}$

Q5 Using $x_{n+1} = \sqrt{(5 - x_n)}$ with $x_1 = 2$, what is the value of x_3 correct to 4 decimal places?

A	B	C	D
1.8077	1.7321	1.7867	1.7913

Q6 What condition on $F'(\alpha)$ guarantees that $x_{n+1} = F(x_n)$ converges for x_1 close to root α?

A	B	C	D				
	$F'(\alpha)$	< 1	$F'(\alpha) > 1$		$F'(\alpha)$	> 1	$F'(\alpha) < -1$

Q7 Which diagram pattern appears when $x_{n+1} = F(x_n)$ converges with $-1 < F'(\alpha) < 0$ near α?

A	B	C	D
A cobweb diagram spiralling alternately around α	A staircase diagram stepping one way towards α	A staircase diagram stepping away from α	A horizontal line with no steps

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Q8 Which diagram pattern appears when $x_{n+1} = F(x_n)$ converges with $0 < F'(\alpha) < 1$ near α?

A	B	C	D
A staircase diagram stepping monotonically towards α	A cobweb diagram spiralling alternately around α	A cobweb diagram spiralling outwards from α	A vertical asymptote at $x = \alpha$

Q9 The sequence $x_{n+1} = (6 - x_n^2)/5$ converges to a positive root α. What is the exact value of α?

A	B	C	D
1	1.2	$\sqrt{6}$	2

Q10 To prove that a root α of continuous $f(x) = 0$ is 1.47 to 2 d.p., where must a sign change be shown?

A	B	C	D
Between $x = 1.465$ and $x = 1.475$	Between $x = 1.46$ and $x = 1.48$	Between $x = 1.40$ and $x = 1.50$	Between $x = 1.470$ and $x = 1.475$

Q11 Using $x_{n+1} = 2 \sin x_n$ (in radians) with $x_1 = 1.9$, what is x_2 correct to 4 decimal places?

A	B	C	D
1.8926	1.9000	0.9463	1.8955

Q12 At a stationary point of $y = e^{2x} - 6x^2$, $dy/dx = 0$ gives $e^{2x} = 6x$. Which formula isolates x from e^{2x}?

A	B	C	D
$x_{n+1} = \frac{1}{2} \ln(6x_n)$	$x_{n+1} = e^{2x_n} / 6$	$x_{n+1} = \ln(3x_n)$	$x_{n+1} = 2 \ln(6x_n)$

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p23-6-numerical-solutions-of-equations/