P23·3

Trigonometry

CHAPTER ASSIGNMENT · 965 MARKS

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given. Show full working and leave exact answers in surd or π form where asked. Give non-exact angles in degrees to 1 decimal place (or α in harmonic form to 2 decimal places where specified) and in radians to 3 significant figures.

Practice questions from *Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook* (Cambridge University Press, Sue Pemberton and Julianne Hughes), Chapter 3 (Trigonometry) and Cross-topic review exercise 1, for classroom use. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Prerequisite knowledge · Check your skills

3.0.1 Sketch the graphs of these functions for $0° \leq x \leq 360°$.

(a) $y = 2 \sin x + 1$ [2]

(b) $y = \tan(x - 90°)$ [2]

(c) $y = \cos(2x + 90°)$ [2]

3.0.2 Find the exact values of:

(a) $\sin 120°$ [1]

(b) cos 225° [1]

(c) tan 150° [1]

3.0.3 Solve for $0° \leq x \leq 360°$.

(a) $5 \sin x = 3 \cos x$ [3]

(b) $2 \cos^2 x - \sin x = 1$ [4]

Exercise 3A · The cosecant, secant and cotangent ratios

3A.1 Find the exact values of:

(a) sec 60° · (b) cosec 45° · (c) cot 120° · (d) sec 300° · (e) cosec 135° · (f) cot 330° · (g) sec 150° · (h) cot(−30°). [8]

3A.2 Find the exact values of:

(a) $\operatorname{cosec}(\pi/6)$ · (b) $\cot(\pi/3)$ · (c) $\sec(\pi/4)$ · (d) $\cot(2\pi/3)$ · (e) $\sec(5\pi/6)$ · (f) $\operatorname{cosec}(7\pi/4)$ · (g) $\cot(4\pi/3)$ · (h) $\sec(-\pi/6)$. [8]

3A.3 Solve each equation for $0° \leq x \leq 360°$.

(a) $\sec x = 3$ [2]

(b) $\cot x = 0.8$ [2]

(c) $\operatorname{cosec} x = -3$ [2]

(d) $3 \sec x - 4 = 0$ [2]

3A.4 Solve each equation for $0 \leq x \leq 2\pi$.

(a) $\operatorname{cosec} x = 2$ [2]

(b) sec x = −1 [2]

(c) cot x = 2 [2]

(d) 2 cot x + 5 = 0 [2]

3A.5 Solve each equation for 0° ≤ x ≤ 180°.

(a) cosec 2x = 1.2 [3]

(b) sec 2x = 4 [3]

(c) cot 2x = 1 [3]

(d) $2 \sec 2x = 3$ [3]

3A.6 Solve each equation for the given domains.

(a) $\operatorname{cosec}(x - 30°) = 2$ for $0° \leq x \leq 360°$ [3]

(b) $\sec(2x + 60°) = -1.5$ for $0° \leq x \leq 180°$ [3]

(c) $\cot(x + \pi/4) = 2$ for $0 \leq x \leq 2\pi$ [3]

(d) $2\,\text{cosec}(2x - 1) = 3$ for $-\pi \leq x \leq \pi$ [4]

3A.7 Solve each equation for $-180° \leq x \leq 180°$.

(a) $\text{cosec}^2 x = 4$ [3]

(b) $\sec^2 x = 9$ [3]

(c) $9\,\cot^2(1/2\,x) = 4$ [3]

(d) $\sec x = \cos x$ [3]

(e) $\operatorname{cosec} x = \sec x$ [3]

(f) $2 \tan x = 3 \operatorname{cosec} x$ [4]

3A.8 Solve each equation for $0° \leq \theta \leq 360°$.

(a) $2 \tan^2\theta - 1 = \sec \theta$ [4]

(b) $3 \operatorname{cosec}^2\theta = 13 - \cot \theta$ [4]

(c) $2\cot^2\theta - \operatorname{cosec}\theta = 13$ [4]

(d) $\operatorname{cosec}\theta + \cot\theta = 2\sin\theta$ [4]

(e) $\tan^2\theta + 3\sec\theta = 0$ [4]

(f) $\sqrt{3}\sec^2\theta = \operatorname{cosec}\theta$ [4]

3A.9 Solve each equation for $0° \leq \theta \leq 180°$.

(a) $\sec \theta = 3 \cos \theta - \tan \theta$ [4]

(b) $2 \sec^2 2\theta = 3 \tan 2\theta + 1$ [4]

(c) $\sec^4 \theta + 2 = 6 \tan^2 \theta$ [4]

(d) $2 \cot^2 2\theta + 7 \operatorname{cosec} 2\theta = 2$ [4]

3A.10 Solve each equation for $0 \leq \theta \leq 2\pi$.

(a) $\tan^2\theta + 3\sec\theta + 3 = 0$ [4]

(b) $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$ [4]

3A.11 (a) Sketch each of the following functions for the interval $0 \leq x \leq 2\pi$.

(i) $y = 1 + \sec x$ [2]

(ii) $y = \cot 2x$ [2]

(iii) $y = 2\operatorname{cosec}(x - \pi/2)$ [2]

(iv) $y = 1 - \sec x$ [2]

(v) $y = 1 + \operatorname{cosec}(1/2\ x)$ [2]

(vi) $y = \sec(2x + \pi/4)$ [2]

(b) Write down the equation of each of the asymptotes for your graph for part (a)(vi). [2]

3A.12 Prove each of these identities.

(a) $\sin x + \cos x \cot x \equiv \operatorname{cosec} x$ [3]

(b) $\operatorname{cosec} x - \sin x \equiv \cos x \cot x$ [3]

(c) $\sec x \operatorname{cosec} x - \cot x \equiv \tan x$ [3]

(d) $(1 + \sec x)(\operatorname{cosec} x - \cot x) \equiv \tan x$ [3]

3A.13 Prove each of these identities.

(a) $1 / (\tan x + \cot x) \equiv \sin x \cos x$ [3]

(b) $\sec^2 x + \sec x \tan x \equiv 1 / (1 - \sin x)$ [3]

(c) $(1 - \cos^2 x) / (\sec^2 x - 1) \equiv 1 - \sin^2 x$ [3]

(d) $(1 + \tan^2 x) / \tan x \equiv \sec x \operatorname{cosec} x$ [3]

(e) $\sin x / (1 - \cos^2 x) \equiv \operatorname{cosec} x$ [3]

(f) $(1 + \sin x) / (1 - \sin x) \equiv (\tan x + \sec x)^2$ [4]

(g) $1 / (1 + \cos x) + 1 / (1 - \cos x) \equiv 2 \operatorname{cosec}^2 x$ [3]

(h) $\cos x / (1 + \sin x) + \cos x / (1 - \sin x) \equiv 2 \sec x$ [3]

3A.14 Solve each equation for $0° \leq \theta \leq 180°$.

(a) $6 \sec^3\theta - 5 \sec^2\theta - 8 \sec \theta + 3 = 0$ [5]

(b) $2 \cot^3\theta + 3 \operatorname{cosec}^2\theta - 8 \cot \theta = 0$ [5]

Explore 3.1 · Discovering the compound angle formulae

E3.1.1 Sunita says that: sin(A + B) = sin A + sin B. Use a calculator with A = 20° and B = 50° to prove that Sunita is wrong. [2]

E3.1.2 In triangle PQR, RX is perpendicular to the base PQ (with X on PQ between P and Q). At vertex R, angle PRX = A and angle QRX = B (where A and B are acute angles). The side lengths are QR = p, PR = q and altitude RX = h.

(a) Write down an expression for h in terms of: (i) q and A · (ii) p and B [2]

(b) Using the formula area of triangle = (1/2)ab sin C, we can write Area of triangle PQR = (1/2)pq sin(A + B). Find an expression in terms of p, q, A and B for the area of: (i) triangle PRX · (ii) triangle QRX [2]

(c) Use your expressions for the areas of triangles PQR, PRX and QRX to prove that: sin(A + B) ≡ sin A cos B + cos A sin B [3]

E3.1.3 In the identity sin(A + B) ≡ sin A cos B + cos A sin B, replace B by −B to show that: sin(A − B) ≡ sin A cos B − cos A sin B [2]

E3.1.4 In the identity sin(A − B) ≡ sin A cos B − cos A sin B, replace A by (90° − A) to show that: cos(A + B) ≡ cos A cos B − sin A sin B [2]

E3.1.5 In the identity cos(A + B) ≡ cos A cos B − sin A sin B, replace B by −B to show that: cos(A − B) ≡ cos A cos B + sin A sin B [2]

E3.1.6 By writing tan(A + B) as sin(A + B) / cos(A + B) and tan(A − B) as sin(A − B) / cos(A − B), show that:

tan(A + B) ≡ (tan A + tan B) / (1 − tan A tan B) and tan(A − B) ≡ (tan A − tan B) / (1 + tan A tan B) [4]

Exercise 3B · Compound angle formulae

3B.1 Expand and simplify cos(x + 30°). [2]

3B.2 Without using a calculator, find the exact value of each of the following:

(a) sin 20° cos 70° + cos 20° sin 70° [2]

(b) sin 172° cos 37° − cos 172° sin 37° [2]

(c) cos 25° cos 35° − sin 25° sin 35° [2]

(d) cos 99° cos 69° + sin 99° sin 69° [2]

(e) (tan 25° + tan 20°) / (1 − tan 25° tan 20°) [2]

(f) (tan 82° − tan 52°) / (1 + tan 82° tan 52°) [2]

3B.3 Without using a calculator, find the exact value of:

(a) sin 75° · (b) tan 75° · (c) cos 105° · (d) tan(−15°) · (e) sin(π/12) · (f) cos(19π/12) · (g) cot(7π/12) · (h) sin(7π/12). [16]

3B.4 Given that cos x = 4/5 and that 0° < x < 90°, without using a calculator find the exact value of cos(x − 60°). [3]

3B.5 Given that sin A = 4/5 and sin B = 5/13, where A and B are acute, show that sin(A + B) = 63/65. [3]

3B.6 Given that sin A = 12/13 and sin B = 3/5, where A is obtuse and B is acute, find the value of:

(a) sin(A + B) [2]

(b) cos(A − B) [2]

(c) tan(A + B) [2]

3B.7 Given that cos A = −4/5 and sin B = −8/17 and that A and B lie in the same quadrant, find the value of:

(a) sin(A + B) [2]

(b) cos(A + B) [2]

(c) tan(A − B) [2]

3B.8 Given that tan A = t and that tan(A − B) = 2, find tan B in terms of t. [3]

3B.9 If cos(A − B) = 3 cos(A + B), find the exact value of tan A tan B. [3]

3B.10 (a) Given that $8 + \text{cosec}^2\theta = 6 \cot \theta$, find the value of tan θ. [3]

(b) Hence find the exact value of tan(θ + 45°). [2]

3B.11 (a) Given that x is acute and that $2 \sec^2 x + 7 \tan x = 17$, find the exact value of tan x. [3]

(b) Hence find the exact value of tan(225° − x). [2]

3B.12 (a) Show that the equation $\sin(x + 30°) = 5\cos(x - 60°)$ can be written in the form $\cot x = -\sqrt{3}$. [3]

(b) Hence solve the equation $\sin(x + 30°) = 5\cos(x - 60°)$ for $-180° < x < 180°$. [2]

3B.13 Solve each equation for $0° \leq x \leq 360°$.

(a) $\cos(x + 30°) = 2\sin x$ [4]

(b) $\cos(x - 60°) = 3\cos x$ [4]

(c) $\sin(30° - x) = 4\sin x$ [4]

(d) $\cos(x + 30°) = 2\sin(x + 60°)$ [4]

3B.14 Solve each equation for $0° \leq x \leq 180°$.

(a) $2\tan(60° - x) = \tan x$ [4]

(b) $2\tan(45° - x) = 3\tan x$ [4]

(c) $\sin(x + 60°) = 2\cos(x + 45°)$ [4]

(d) $\sin(x - 45°) = 2\cos(x + 60°)$ [4]

(e) $\tan(x + 45°) = 6\tan x$ [4]

(f) $\cos(x + 225°) = 2\sin(x - 60°)$ [4]

3B.15 Solve the equation $\tan(x - 45°) + \cot x = 2$ for $0° \leq x \leq 180°$. [5]

3B.16 Use the expansions of $\cos(5x + x)$ and $\cos(5x - x)$ to prove that $\cos 6x + \cos 4x \equiv 2\cos 5x \cos x$. [3]

3B.17 Given that $\sin x + \sin y = p$ and $\cos x + \cos y = q$, find an expression for $\cos(x - y)$, in terms of p and q. [4]

Exercise 3C · Double angle formulae

3C.1 Express each of the following as a single trigonometric ratio.

(a) $2 \sin 28° \cos 28°$ [1]

(b) $2 \cos^2 34° - 1$ [1]

(c) $2 \tan 17° / (1 - \tan^2 17°)$ [1]

3C.2 Given that $\tan x = 4/3$, where $0° < x < 90°$, find the exact value of:

(a) $\sin 2x$ · (b) $\cos 2x$ · (c) $\tan 2x$ · (d) $\tan 3x$ [8]

3C.3 Given that $\cos 2x = -527/625$ and that $0° < x < 90°$, find the exact value of:

(a) $\sin 2x$ · (b) $\tan 2x$ · (c) $\cos x$ · (d) $\tan x$ [8]

3C.4 Given that $\cos x = -3/5$ and that $90° < x < 180°$, find the exact value of:

(a) $\sin 2x$ · (b) $\sin 4x$ · (c) $\tan 2x$ · (d) $\tan(1/2\ x)$ [8]

3C.5 Given that $3 \cos 2x + 17 \sin x = 8$, find the exact value of $\sin x$. [4]

3C.6 Solve each equation for $0° \leq \theta \leq 360°$.

(a) $2 \sin 2\theta = \cos \theta$ [4]

(b) $2 \cos 2\theta + 3 = 4 \cos \theta$ [4]

(c) $2 \cos 2\theta + 1 = \sin \theta$ [4]

3C.7 Solve each equation for $0° \leq \theta \leq 180°$.

(a) $2 \sin 2\theta \tan \theta = 1$ [4]

(b) $3 \cos 2\theta + \cos \theta = 2$ [4]

(c) $2 \operatorname{cosec} 2\theta + 2 \tan \theta = 3 \sec \theta$ [4]

(d) $2 \tan 2\theta = 3 \cot \theta$ [4]

(e) $\tan 2\theta = 4 \cot \theta$ [4]

(f) $\cot 2\theta + \cot \theta = 3$ [4]

3C.8 Express $\cos^2 2x$ in terms of $\cos 4x$. [2]

3C.9 (a) Use the expansion of $\cos(2x + x)$ to show that $\cos 3x \equiv 4\cos^3 x - 3\cos x$. [3]

(b) Express $\sin 3x$ in terms of $\sin x$. [3]

3C.10 Solve $\tan 2\theta + 2\tan\theta = 3\cot\theta$ for $0° \leq \theta \leq 180°$. [5]

3C.11 (a) Prove that $\tan\theta + \cot\theta \equiv 2 / \sin 2\theta$. [3]

(b) Hence find the exact value of $\tan(\pi/12) + \cot(\pi/12)$. [2]

3C.12 (a) Prove that $2\,\text{cosec}\,2\theta\tan\theta \equiv \sec^2\theta$. [3]

(b) Hence solve the equation $\text{cosec}\,2\theta\tan\theta = 2$ for $-\pi < \theta < \pi$. [4]

3C.13 (a) Prove that $\cos 4x + 4\cos 2x \equiv 8\cos^4 x - 3$. [4]

(b) Hence solve the equation $2\cos 4x + 8\cos 2x = 3$ for $-\pi \leq x \leq \pi$. [4]

3C.14 (a) Show that $\theta = 18°$ is a root of the equation $\sin 3\theta = \cos 2\theta$. [1]

(b) Express $\sin 3\theta$ and $\cos 2\theta$ in terms of $\sin \theta$. [3]

(c) Hence show that $\sin 18°$ is a root of the equation $4x^3 - 2x^2 - 3x + 1 = 0$. [2]

(d) Hence find the exact value of $\sin 18°$. [3]

3C.15 Find the range of values of θ between 0 and 2π for which $\cos 2\theta > \cos \theta$. [5]

3C.16 Solve the inequality $\cos 2\theta - 3\sin\theta - 2 \geq 0$ for $0° \leq \theta \leq 360°$. [5]

3C.17 In triangle ABC, side BC = a, side AC = b, angle ABC = x and angle ACB = 3x. Prove that $a = 4b\cos 2x \cos x$. [4]

3C.18 (a) Use the expansions of $\cos(2x + x)$ and $\cos(2x - x)$ to prove that: $\cos 3x + \cos x \equiv 2\cos 2x \cos x$ [3]

(b) Solve $\cos 3x + \cos 2x + \cos x > 0$ for $0° < x < 360°$. [5]

3C.19 Solve the inequality $\cos 4\theta + 3\cos 2\theta + 1 < 0$ for $0° \leq \theta \leq 360°$. [5]

Explore 3.2 · Odd one out

E3.2.1 Find the trigonometric expression from the following seven that does not match the other six expressions, and state what the matching six simplify to:

(i) $(1 - \cos 2x) / \sin 2x$ · (ii) $\operatorname{cosec} 2x - \cot 2x$ · (iii) $(\sin 2x + \sin x) / (\cos 2x + \cos x + 1)$ · (iv) $1 / (\operatorname{cosec} 2x + \cot 2x)$ · (v) $\sin 2x / (1 + \cos 2x)$ · (vi) $2 / (\sec^2 x \sin 2x)$ · (vii) $\cot x - 2 \cot 2x$. [4]

Exercise 3D · Further trigonometric identities

3D.1 Prove each of these identities.

(a) $\tan A + \cot A \equiv 2 \operatorname{cosec} 2A$ [3]

(b) $1 - \tan^2 A \equiv \cos 2A \sec^2 A$ [3]

(c) $\tan 2A - \tan A \equiv \tan A \sec 2A$ [3]

(d) $(\cos A + 3 \sin A)^2 \equiv 5 - 4 \cos 2A + 3 \sin 2A$ [3]

(e) $\cot 2A + \tan A \equiv (1/2) \operatorname{cosec} A \sec A$ [3]

(f) $\operatorname{cosec} 2A + \cot 2A \equiv \cot A$ [3]

(g) $\sec^2 A \operatorname{cosec}^2 A \equiv 4 \operatorname{cosec}^2 2A$ [3]

(h) $\sin(A + B) \sin(A - B) \equiv \sin^2 A - \sin^2 B$ [3]

3D.2 Prove each of these identities.

(a) $\sin A / \cos B + \cos A / \sin B \equiv 2\cos(A - B) / \sin 2B$ [3]

(b) $(\cos A + \sin A) / (\cos A - \sin A) \equiv \sec 2A + \tan 2A$ [4]

(c) $(1 - \tan^2 A) / (1 + \tan^2 A) \equiv \cos 2A$ [3]

(d) $(\cos 2A + \sin 2A - 1) / (\cos 2A - \sin 2A + 1) \equiv \tan A$ [4]

(e) $(\sin 3A + \sin A) / (2 \sin 2A) \equiv \cos A$ [4]

(f) $(\cos 3A - \sin 3A) / (1 - 2 \sin 2A) \equiv \cos A + \sin A$ [4]

(g) $(\cos 2A + 9 \cos A + 5) / (4 + \cos A) \equiv 2 \cos A + 1$ [3]

(h) $(\cos^3 A - \sin^3 A) / (\cos A - \sin A) \equiv (2 + \sin 2A) / 2$ [3]

3D.3 Use the fact that $4A = 2 \times 2A$ to show that:

(a) $\sin 4A / \sin A \equiv 8 \cos^3 A - 4 \cos A$ [3]

(b) $\cos 4A + 4\cos 2A \equiv 8\cos^4 A - 3$ [3]

3D.4 Prove the identity $8\sin^2 x \cos^2 x \equiv 1 - \cos 4x$. [3]

3D.5 Prove the identity $(2\sin A + \cos A)^2 \equiv (1/2)(4\sin 2A - 3\cos 2A + 5)$. [3]

3D.6 Use the expansions of $\cos(3x - x)$ and $\cos(3x + x)$ to prove the identity: $\cos 2x - \cos 4x \equiv 2\sin 3x \sin x$. [3]

Explore 3.3 & 3.4 · Harmonic form derivations

E3.3.1 Consider the function $y = 3\sin\theta + 4\cos\theta$ for $-90° \le \theta \le 360°$, which is a single wave oscillating between -5 and 5.

(a) What are the amplitude and period of this graph? [2]

(b) Explain how this graph can be obtained by transforming the graph of: (i) $y = \sin\theta$ · (ii) $y = \cos\theta$ [2]

(c) Write the function $y = 3\sin\theta + 4\cos\theta$ in the form: (i) $y = R\sin(\theta + \alpha)$ · (ii) $y = R\cos(\theta - \beta)$, giving R and the acute angles α and β to 1 decimal place. [3]

E3.4.1 By expanding the right-hand side and equating coefficients of $\sin\theta$ and $\cos\theta$, show that for $R > 0$ and $0° < \alpha < 90°$:

(a) $a\sin\theta - b\cos\theta \equiv R\sin(\theta - \alpha)$ where $R = \sqrt{(a^2 + b^2)}$ and $\tan\alpha = b/a$ [2]

(b) $a\cos\theta + b\sin\theta \equiv R\cos(\theta - \alpha)$ where $R = \sqrt{(a^2 + b^2)}$ and $\tan\alpha = b/a$ [2]

(c) $a\cos\theta - b\sin\theta \equiv R\cos(\theta + \alpha)$ where $R = \sqrt{(a^2 + b^2)}$ and $\tan\alpha = b/a$ [2]

Exercise 3E · Expressing $a\sin\theta + b\cos\theta$ in the form $R\sin(\theta \pm \alpha)$ or $R\cos(\theta \pm \alpha)$

3E.1 (a) Express $15\sin\theta - 8\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $15\sin\theta - 8\cos\theta = 10$ for $0° < \theta < 360°$. [3]

3E.2 (a) Express $2\cos\theta - 3\sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $2\cos\theta - 3\sin\theta = 1.3$ for $0° < \theta < 360°$. [3]

3E.3 (a) Express $15\sin\theta - 8\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $15\sin\theta - 8\cos\theta = 3$ for $0° < \theta < 360°$. [3]

(c) Find the greatest value of $30\sin\theta - 16\cos\theta$ as θ varies. [1]

3E.4 (a) Express $4\sin\theta - 6\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $4\sin\theta - 6\cos\theta = 3$ for $0° < \theta < 180°$. [3]

(c) Find the greatest and least values of $(4 \sin \theta - 6 \cos \theta)^2 - 3$ as θ varies. [2]

3E.5 (a) Express $3 \sin \theta + 4 \cos \theta$ in the form $R \sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $3 \sin \theta + 4 \cos \theta = 2$ for $0° < \theta < 360°$. [3]

(c) Find the least value of $3 \sin \theta + 4 \cos \theta + 3$ as θ varies. [1]

3E.6 (a) Express $\cos \theta + \sqrt{3} \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \pi/2$. Give the exact values of R and α. [3]

(b) Hence prove that $1 / (\cos \theta + \sqrt{3} \sin \theta)^2 \equiv (1/4) \sec^2(\theta - \pi/3)$. [2]

3E.7 (a) Express $8 \sin 2\theta + 4 \cos 2\theta$ in the form $R \sin(2\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $8 \sin 2\theta + 4 \cos 2\theta = 3$ for $0° < \theta < 360°$. [4]

(c) Find the least value of $10 / (8 \sin 2\theta + 4 \cos 2\theta)^2$ as θ varies. [2]

3E.8 (a) Express $\cos \theta - \sqrt{2} \sin \theta$ in the form $R \cos(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $\cos \theta - \sqrt{2} \sin \theta = -1$ for $0° < \theta < 360°$. [3]

(c) Find the least value of $1 / (\sqrt{2} \cos \theta - 2 \sin \theta)^2$ as θ varies. [2]

3E.9 (a) Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0 < \alpha < \pi/2$. Give the exact values of R and α. [3]

(b) Show that one solution of the equation $\cos\theta - \sin\theta = (1/2)\sqrt{6}$ is $\theta = 19\pi/12$ and find the other solution in the interval $0 < \theta < 2\pi$. [3]

(c) State the set of values of k for which the equation $\cos\theta - \sin\theta = k$ has any solutions. [1]

3E.10 (a) Express $\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(b) Hence solve the equation $\sin\theta - 3\cos\theta = -2$ for $0° < \theta < 360°$. [3]

(c) Find the greatest possible value of $1 + \sin 2\theta - 3\cos 2\theta$ as θ varies and determine the smallest positive value of θ for which this greatest value occurs. [3]

3E.11 (a) Express $\sqrt{5}\cos\theta + 2\sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(b) Find the smallest positive angle θ that satisfies the equation $\sqrt{5}\cos\theta + 2\sin\theta = 3$. [2]

(c) Find the smallest positive angle θ that satisfies the equation $\sqrt{5}\cos(1/2\,\theta) + 2\sin(1/2\,\theta) = -1$. [3]

3E.12 (a) Given that $3\sec\theta + 4\operatorname{cosec}\theta = 2\operatorname{cosec}2\theta$, show that $3\sin\theta + 4\cos\theta = 1$. [3]

(b) Express $3\sin\theta + 4\cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the value of α correct to 2 decimal places. [3]

(c) Hence solve the equation $3\sec\theta + 4\operatorname{cosec}\theta = 2\operatorname{cosec}2\theta$ for $0° < \theta < 360°$. [3]

3E.13 (a) Express $\sin(\theta + 30°) + \cos\theta$ in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact values of R and α. [4]

(b) Hence solve the equation $\sin(\theta + 30°) + \cos\theta = 1$ for $0° < \theta < 360°$. [3]

3E.14 (a) Find the maximum and minimum values of $7\sin^2\theta + 9\cos^2\theta + 4\sin\theta\cos\theta + 2$. [5]

(b) Hence, or otherwise, solve the equation $7\sin^2\theta + 9\cos^2\theta + 4\sin\theta\cos\theta = 10$ for $0° < \theta < 360°$. [4]

End-of-chapter review exercise 3

R3.1 Sketch the graph of $y = 3\sec(2x - 90°)$ for $0° < x < 180°$. [3]

R3.2 By expressing the equation $\operatorname{cosec}\theta = 3\sin\theta + \cot\theta$ in terms of $\cos\theta$ only, solve the equation for $0° < \theta < 180°$. [5]

R3.3 Given that cos A = 1/4, where 270° < A < 360°, find the exact value of sin 2A. [5]

R3.4 Solve the equation $2\tan^2 x + \sec x = 1$ for $0° \leq x \leq 360°$. [6]

R3.5 Solve the equation $2\cot^2 x + 5\operatorname{cosec} x = 10$ for $0° < x < 360°$. [6]

R3.6 (a) Prove that $\sin(x + 60°) + \cos(x + 30°) \equiv \sqrt{3}\cos x$. [3]

(b) Hence solve the equation $\sin(x + 60°) + \cos(x + 30°) = 3/2$ for $0° < x < 360°$. [3]

R3.7 (a) Prove that $\sin(60° - x) + \cos(30° - x) \equiv \sqrt{3} \cos x$. [3]

(b) Hence solve the equation $\sin(60° - x) + \cos(30° - x) = (2/5) \sec x$ for $0° < x < 360°$. [3]

R3.8 (i) Show that the equation $\tan(x + 45°) = 6 \tan x$ can be written in the form $6 \tan^2 x - 5 \tan x + 1 = 0$. [3]

(ii) Hence solve the equation $\tan(x + 45°) = 6 \tan x$, for $0° < x < 180°$. [3]

R3.9 (a) Prove that $\tan(x + 45°) - \tan(45° - x) \equiv 2 \tan 2x$. [4]

(b) Hence solve the equation $\tan(x + 45°) - \tan(45° - x) = 6$ for $0° < x < 180°$. [3]

R3.10 (i) Express $3\cos\theta + \sin\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $3\cos 2x + \sin 2x = 2$, giving all solutions in the interval $0° \leq x \leq 360°$. [5]

R3.11 (a) Prove that $\cos(60° - x) + \cos(300° - x) \equiv \cos x$. [3]

(b) Hence:

(i) find the exact value of $\cos 15° + \cos 255°$ [2]

(ii) solve the equation $\cos(60° - x) + \cos(300° - x) = (1/4)\operatorname{cosec} x$ for $0° < x < 180°$. [3]

R3.12 (a) Prove the identity $(2 \sin 2\theta - 3 \cos 2\theta + 3) / \sin \theta \equiv 4 \cos \theta + 6 \sin \theta$. [3]

(b) Express $4 \cos \theta + 6 \sin \theta$ in the form $R \cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \pi/2$. Give the exact value of R and the value of α correct to 2 decimal places. [4]

(c) Write down the greatest value of $((2 \sin 2\theta - 3 \cos 2\theta + 3) / \sin \theta)^2$. [1]

R3.13 (i) Express $4 \sin \theta - 6 \cos \theta$ in the form $R \sin(\theta - \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Solve the equation $4 \sin \theta - 6 \cos \theta = 3$ for $0° \leq \theta \leq 360°$. [4]

(iii) Find the greatest and least possible values of $(4 \sin \theta - 6 \cos \theta)^2 + 8$ as θ varies. [2]

R3.14 (i) By first expanding $\sin(2\theta + \theta)$, show that $\sin 3\theta = 3 \sin \theta - 4 \sin^3\theta$. [4]

(ii) Show that, after making the substitution $x = 2 \sin \theta / \sqrt{3}$, the equation $x^3 - x + (1/6)\sqrt{3} = 0$ can be written in the form $\sin 3\theta = 3/4$. [1]

(iii) Hence solve the equation $x^3 - x + (1/6)\sqrt{3} = 0$, giving your answers correct to 3 significant figures. [4]

R3.15 (a) Prove the identity $1 / (\sin(x + 30°) + \cos(x + 60°)) \equiv \sec x$. [3]

(b) Hence solve the equation $2 / (\sin(x + 30°) + \cos(x + 60°)) = 7 - \tan^2 x$ for $0° < x < 360°$. [6]

R3.16 (a) Prove the identity $\text{cosec}^4 x - \cot^4 x \equiv \text{cosec}^2 x + \cot^2 x$. [3]

(b) Hence solve the equation $\text{cosec}^4 x - \cot^4 x = 16 - \cot x$ for $0° < x < 180°$. [6]

R3.17 The curves C_1 and C_2 have equations $y = 1 + 4\cos 2x$ and $y = 2\cos^2 x - 4\sin 2x$.

(a) Show that the x-coordinates of the points where C_1 and C_2 intersect satisfy the equation $3\cos 2x + 4\sin 2x = 0$. [3]

(b) Express $3\cos 2x + 4\sin 2x$ in the form $R\sin(2x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give the exact value of R and the value of α correct to 2 decimal places. [3]

(c) Hence find all the roots of the equation $3\cos 2x + 4\sin 2x = 0$ for $0° < x < 180°$. [3]

Cross-topic review exercise 1 (Chapters 1–3)

CT1.1 Find the set of values of x satisfying the inequality $3|x - 1| < |2x + 1|$. [4]

CT1.2 Solve the equation $2|3^x - 1| = 3^x$, giving your answers correct to 3 significant figures. [4]

CT1.3 (i) Solve the equation $|3x + 4| = |3x - 11|$. [3]

(ii) Hence, using logarithms, solve the equation $|3 \times 2^y + 4| = |3 \times 2^y - 11|$, giving the answer correct to 3 significant figures. [2]

CT1.4 (i) Solve the equation $2|x - 1| = 3|x|$. [3]

(ii) Hence solve the equation $2|5^x - 1| = 3|5^x|$, giving your answer correct to 3 significant figures. [2]

CT1.5 The variables x and y satisfy the equation $y = A(b^x)$, where A and b are constants. The graph of ln y against x is a straight line passing through the points (0, 2.14) and (5, 4.49). Find the values of A and b, correct to 1 decimal place. [5]

CT1.6 (i) It is given that x satisfies the equation $3^{2x} = 5(3^x) + 14$. Find the value of 3^x and, using logarithms, find the value of x correct to 3 significant figures. [4]

(ii) Hence state the values of x satisfying the equation $3^{2|x|} = 5(3^{|x|}) + 14$. [1]

CT1.7 The variables x and y satisfy the equation $x^n y = C$, where n and C are constants. When x = 1.10, y = 5.20, and when x = 3.20, y = 1.05.

(i) Find the values of n and C. [5]

(ii) Explain why the graph of ln y against ln x is a straight line. [1]

CT1.8 Given that $3e^x + 8e^{-x} = 14$, find the possible values of e^x and hence solve the equation $3e^x + 8e^{-x} = 14$ correct to 3 significant figures. [6]

CT1.9 The angles θ and φ lie between 0° and 180°, and are such that $\tan(\theta - \varphi) = 3$ and $\tan\theta + \tan\varphi = 1$. Find the possible values of θ and φ. [6]

CT1.10 (i) Solve the equation $|4x - 1| = |x - 3|$. [3]

(ii) Hence solve the equation $|4^{y+1} - 1| = |4^y - 3|$ correct to 3 significant figures. [3]

CT1.11 The polynomial $4x^3 + ax^2 + 9x + 9$, where a is a constant, is denoted by p(x). It is given that when p(x) is divided by $(2x - 1)$ the remainder is 10.

(i) Find the value of a and hence verify that $(x - 3)$ is a factor of p(x). [3]

(ii) When a has this value, solve the equation $p(x) = 0$. [4]

CT1.12 The polynomial $2x^3 - 4x^2 + ax + b$, where a and b are constants, is denoted by p(x). It is given that when p(x) is divided by (x + 1) the remainder is 4, and that when p(x) is divided by (x − 3) the remainder is 12.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, find the quotient and remainder when p(x) is divided by $(x^2 - 2)$. [3]

CT1.13 (i) Express cos x + 3 sin x in the form R cos(x − α), where R > 0 and 0° < α < 90°, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation cos 2θ + 3 sin 2θ = 2, for 0° < θ < 90°. [5]

CT1.14 It is given that $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$.

(i) Show that $16x^3 - 24x^2 - 15x - 2 = 0$. **[3]**

(ii) By first using the factor theorem, factorise $16x^3 - 24x^2 - 15x - 2$ completely. **[4]**

(iii) Hence solve the equation $2\ln(4x - 5) + \ln(x + 1) = 3\ln 3$. **[1]**

CT1.15 The polynomial $8x^3 + ax^2 + bx - 1$, where a and b are constants, is denoted by p(x). It is given that $(x + 1)$ is a factor of p(x) and that when p(x) is divided by $(2x + 1)$ the remainder is 1.

(i) Find the values of a and b. **[5]**

(ii) When a and b have these values, factorise p(x) completely. [3]

CT1.16 The polynomial $3x^3 + 2x^2 + ax + b$, where a and b are constants, is denoted by p(x). It is given that (x – 1) is a factor of p(x), and that when p(x) is divided by (x – 2) the remainder is 10.

(i) Find the values of a and b. [5]

(ii) When a and b have these values, solve the equation p(x) = 0. [4]

CT1.17 (i) The polynomial $x^3 + ax^2 + bx + 8$, where a and b are constants, is denoted by p(x). It is given that when p(x) is divided by (x − 3) the remainder is 14, and that when p(x) is divided by (x + 2) the remainder is 24. Find the values of a and b. [5]

(ii) When a and b have these values, find the quotient when p(x) is divided by $x^2 + 2x - 8$ and hence solve the equation p(x) = 0. [4]

CT1.18 (i) Given that (x + 2) and (x + 3) are factors of $5x^3 + ax^2 + b$, find the values of the constants a and b. [4]

(ii) When a and b have these values, factorise $5x^3 + ax^2 + b$ completely, and hence solve the equation $5^{3y+1} + a \times 5^{2y} + b = 0$, giving any answers correct to 3 significant figures. [5]

CT1.19 (i) Find the quotient and remainder when $x^4 + x^3 + 3x^2 + 12x + 6$ is divided by $(x^2 - x + 4)$. [4]

(ii) It is given that, when $x^4 + x^3 + 3x^2 + px + q$ is divided by $(x^2 - x + 4)$, the remainder is zero. Find the values of the constants p and q. [2]

(iii) When p and q have these values, show that there is exactly one real value of x satisfying the equation $x^4 + x^3 + 3x^2 + px + q = 0$ and state what that value is. [3]

CT1.20 The angle α lies between 0° and 90° and is such that $2\tan^2\alpha + \sec^2\alpha = 5 - 4\tan\alpha$.

(i) Show that $3\tan^2\alpha + 4\tan\alpha - 4 = 0$ and hence find the exact value of $\tan\alpha$. [4]

(ii) It is given that the angle β is such that $\cot(\alpha + \beta) = 6$. Without using a calculator, find the exact value of $\cot\beta$. [5]

CT1.21 (i) Express $5 \sin 2\theta + 2 \cos 2\theta$ in the form $R \sin(2\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$, giving the exact value of R and the value of α correct to 2 decimal places. [3]

(ii) Hence solve the equation $5 \sin 2\theta + 2 \cos 2\theta = 4$, giving all solutions in the interval $0° \leq \theta \leq 360°$. [5]

(iii) Determine the least value of $1 / (10 \sin 2\theta + 4 \cos 2\theta)^2$ as θ varies. [2]

CT1.22 The polynomial p(x) is defined by $p(x) = ax^3 + 3x^2 + bx + 12$, where a and b are constants. It is given that $(x + 3)$ is a factor of p(x). It is also given that the remainder is 18 when p(x) is divided by $(x + 2)$.

(i) Find the values of a and b. [5]

(ii) When a and b have these values:

(a) show that the equation $p(x) = 0$ has exactly one real root, [4]

(b) solve the equation $p(\sec y) = 0$ for $-180° < y < 180°$. [3]

Check yourself

CY.1 Without a calculator, find the exact value of $\sec^2 150° + \operatorname{cosec}^2 225°$. [3]

CY.2 Solve $2\sec^2\theta - 5\tan\theta = 5$ for $0° \leq \theta \leq 360°$. [5]

CY.3 Without using a calculator, find the exact value of $\tan 105°$ in the form $a + b\sqrt{3}$. [4]

CY.4 Prove the identity $(\sin 2\theta + \sin \theta) / (1 + \cos \theta + \cos 2\theta) \equiv \tan \theta$. [4]

CY.5 Express $5 \sin \theta - 12 \cos \theta$ in the form $R \sin(\theta - \alpha)$ with $R > 0$ and $0° < \alpha < 90°$, and hence solve $5 \sin \theta - 12 \cos \theta = 6.5$ for $0° \leq \theta \leq 360°$. [6]

CY.6 State the maximum and minimum values of $20 / (5 \sin \theta - 12 \cos \theta + 17)$ and the values of θ in $0° \leq \theta \leq 360°$ at which they occur. [4]

Chapter quiz

Q1 What is the exact value of cosec 240°?

A	B	C	D
$-2/\sqrt{3}$	$2/\sqrt{3}$	-2	$-\sqrt{3}/2$

Q2 What is the exact value of $\sec(5\pi/6)$?

A	B	C	D
$-2/\sqrt{3}$	$2/\sqrt{3}$	-2	2

Q3 Which Pythagorean identity is true for all θ where the functions are defined?

A	B	C	D
$1 + \tan^2\theta \equiv \sec^2\theta$	$1 - \tan^2\theta \equiv \sec^2\theta$	$\tan^2\theta - 1 \equiv \sec^2\theta$	$1 + \sec^2\theta \equiv \tan^2\theta$

Q4 Solve $\sec^2 x - \tan x - 3 = 0$ for $0° \leq x \leq 180°$.

A	B	C	D
63.4° and 135°	45° and 63.4°	116.6° and 135°	45° and 116.6°

Q5 Simplify $\cos 42° \cos 12° + \sin 42° \sin 12°$ and find its exact value.

A	B	C	D
$\sqrt{3}/2$	1/2	$\sqrt{2}/2$	$(\sqrt{3}+1)/(2\sqrt{2})$

Q6 What is the exact value of $\sin 75°$?

A	B	C	D
$(\sqrt{3}+1)/(2\sqrt{2})$	$(\sqrt{3}-1)/(2\sqrt{2})$	$\sqrt{3}/2 + 1/2$	$\sqrt{2}/2$

Q7 Given $\tan A = t$ and $\tan(A - B) = 2$, express $\tan B$ in terms of t.

A	B	C	D
$(t-2)/(1+2t)$	$(t+2)/(1-2t)$	$(2-t)/(1+2t)$	$(t-2)/(1-2t)$

Q8 Given $\sin x = -3/5$ and $180° < x < 270°$, what is the exact value of $\sin 2x$?

A	B	C	D
24/25	−24/25	7/25	−7/25

Q9 Which expression is NOT identical to $\cos 2A$?

A	B	C	D
$\sin^2 A - \cos^2 A$	$\cos^2 A - \sin^2 A$	$2\cos^2 A - 1$	$1 - 2\sin^2 A$

Q10 Solve $\sin 2x = \sin x$ for $0° \leq x \leq 180°$.

A	B	C	D
0°, 60°, 180°	60° only	0°, 30°, 180°	30° and 150°

Q11 When $3\sin\theta + 4\cos\theta$ is written as $R\sin(\theta + \alpha)$ with $R > 0$ and $0° < \alpha < 90°$, what are R and $\tan\alpha$?

A	B	C	D
$R = 5$, $\tan\alpha = 4/3$	$R = 5$, $\tan\alpha = 3/4$	$R = 7$, $\tan\alpha = 4/3$	$R = \sqrt{7}$, $\tan\alpha = 3/4$

Q12 What are the maximum and minimum values of $15\sin\theta - 8\cos\theta$ as θ varies?

A	B	C	D
Max 17, min −17	Max 23, min −23	Max 15, min −8	Max 289, min −289

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p23-3-trigonometry/