P23·11

Complex numbers

CHAPTER ASSIGNMENT · 420 MARKS

Name _____ Class _____ No. _____ Date _____

This exercise covers material across Pure Mathematics 3 (Vectors, Differential Equations, and Complex Numbers).

Practice questions from *Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook* (Cambridge University Press, Sue Pemberton and Julianne Hughes), Chapter 11 (Complex numbers), Exercises 11A to 11E, End-of-chapter review exercise 11 and Cross-topic review exercise 4, for classroom use. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 11A · Imaginary numbers

11A.1 Write the following numbers in their simplest form: (a) $\sqrt{-144}$ · (b) $\sqrt{(-36/81)}$ · (c) $\sqrt{-90}$ · (d) $\sqrt{-16} + \sqrt{-81}$

11A.2 Simplify: (a) $-5i^2 + 3i^8$ · (b) $9i - (i\sqrt{2})^3$ · (c) $\sqrt{((100i^4 - 16i^2)/4)}$ · (d) $-5/(6i^2)$

11A.3 Solve: (a) $x^2 + 64/25 = 0$ · (b) $4x^2 + 7 = 0$ · (c) $12x^2 + 3 = 0$

Exercise 11B · Complex numbers

11B.1 $z_1 = 5 - 3i$ and $z_2 = 1 + 2i$. Write each of the following in the form $x + iy$, where x and y are real values: (a) $z_1 + z_2^*$ · (b) $z_1^* - z_2$ · (c) $z_1 z_2$ · (d) z_1/z_2

11B.2 Solve these equations: (a) $z^2 + 2z + 13 = 0$ · (b) $z^2 + 4z + 5 = 0$ · (c) $2z^2 - 2z + 5 = 0$ · (d) $z^2 - 6z + 15 = 0$ · (e) $3z^2 + 8z + 10 = 0$ · (f) $2z^2 + 5z + 4 = 0$

11B.3 Find the value of x and the value of y: (a) $(x + 2y) + i(3x - y) = 1 + 10i$ · (b) $(x + y - 4) + 2xi = (5 - y)i$ · (c) $(x - y) + (2x - y)i = -1$

11B.4 Write each of the following in the form x + iy, where x and y are real values: (a) $(1 + 3i)(2 - i)$ · (b) $(4 - 5i)(4 + 5i)$ · (c) $(7 - 3i)^2$ · (d) $(3 - i)^4$ · (e) $(17 - i)/(3 + i)$ · (f) $(-1 + 11i)/(6 - 5i)$ · (g) $13(1 + i)/(2 + 3i)$ · (h) $(3 - 2i)^2/(5 + i)$

11B.5 (a) Show that $z = 1/5 + (2\sqrt{6}/5)i$ is a solution of the equation $5z^2 - 2z + 5 = 0$. (b) Write down the other solution of the equation.

11B.6 Find the quadratic equations that have the following roots: (a) $\alpha = -7i$, $\beta = 7i$ · (b) $\alpha = 1 + 5i$, $\beta = 1 - 5i$ · (c) $\alpha = 2 - 3i$, $\beta = 2 + 3i$ · (d) $\alpha = -5/2 - (\sqrt{31}/2)i$, $\beta = -5/2 + (\sqrt{31}/2)i$

11B.7 Find the complex number z satisfying the equation $z + 4 = 3iz^*$. Give your answer in the form $x + iy$, where x and y are real.

11B.8 Find the complex number u, given that $u(3 - 5i) = 13 + i$.

11B.9 $z = 5 + i\sqrt{3}$ is a root of a quadratic equation. Find this quadratic equation.

11B.10 In an electrical circuit, the voltage (volts), current (amperes) and impedance (ohms) are related by the equation: voltage = current × impedance. The voltage in a particular circuit is 240 V and the impedance is 48 + 36i ohms. Find the current.

Exercise 11C · The complex plane

11C.1 $u = 5 - 2i$. (a) On an Argand diagram, show the points A, B and C representing the complex numbers u, u^* and $-u$, respectively. (b) The points ABCD form a rectangle. Write down the complex number represented by the point D.

11C.2 $z_1 = 1 + 5i$ and $z_2 = -7 - i$. (a) On an Argand diagram, show the points P and Q representing the complex numbers z_1 and z_2, respectively. (b) Write down the complex number represented by the point R, the midpoint of PQ.

11C.3 Find the modulus and argument of each of the following complex numbers: (a) $-12 + 5i$ · (b) $5i$ · (c) $8 + 15i$ · (d) $60 - 11i$ · (e) $-9 - 40i$ · (f) $-1 - i\sqrt{3}$ · (g) $\sqrt{5} + 2i$ · (h) $-24 - 7i$ · (i) $k(1 - i)$, where $k > 0$

11C.4 Points A(−1, 3), B(3, 1) and C(1, −3) are shown on an Argand diagram. (a) Write each of the complex numbers shown in the Argand diagram in modulus-argument form. (b) Show that ABC is a right-angled, isosceles triangle.

11C.5 Write each of these complex numbers in Cartesian form: (a) $3(\cos(\pi/3) + i\sin(\pi/3))$ · (b) $5(\cos(3\pi/8) + i\sin(3\pi/8))$ · (c) $(1/2)e^{i\pi}$ · (d) $3e^{-i\pi/4}$

11C.6 Given that $z(4 - 9i) = 8 + 3i$, find the value of $|z^2|$ and the value of $\arg z^2$.

11C.7 $w = 5(\cos(\pi/6) + i\sin(\pi/6))$ and $z = (3 - 7i)/(5 - 2i)$. (a) Write w in exponential form. (b) Write z in exponential form. (c) Find and simplify an expression for z/w.

11C.8 The complex number $a + bi$ is denoted by z. Given that $|z| = 5$ and $\arg z = \pi/6$, find the value of a and the value of b.

11C.9 Given that $z = r(\cos\theta + i\sin\theta)$, find and simplify an expression for: (a) zz^* · (b) z/z^*

11C.10 $z = r(\cos\theta + i\sin\theta)$. (a) Show that $z^2 = r^2(\cos 2\theta + i\sin 2\theta)$. (b) Show that $z - 1/z = i(2r\sin\theta)$ [when $r = 1$; for general r, $z - r^2/z = i(2r\sin\theta)$].

Exercise 11D · Solving equations

11D.1 (a) Given that $z_1 = -i$ is a root of the equation $z^3 + z^2 + z + k = 0$, find the value of the constant k. (b) Write down the other complex root of the equation and find the third root, stating whether it is real or complex.

11D.2 Solve: (a) $(z - 5)^3 = 8$ · (b) $(2z + 3)^3 = 1/64$

11D.3 Find the roots of the equation $2z^2 + z + 3 = 0$, giving your answer in modulus-argument form.

11D.4 It is given that $z - 3$ is a factor of $z^3 - 3z^2 + 25z - 75$. Solve $z^3 - 3z^2 + 25z - 75 = 0$.

11D.5 $(x + iy)^2 = 55 + 48i$. Find the value of x and the value of y, when x and y are real and positive.

11D.6 It is given that $2z + 1$ is a factor of $2z^3 - 11z^2 + 14z + 10$. Solve $2z^3 - 11z^2 + 14z + 10 = 0$.

11D.7 Given that $z = 3i$ is a root, solve $z^4 - 2z^3 + 14z^2 - 18z + 45 = 0$.

11D.8 Find the square roots of: (a) $24 - 10i$ · (b) $7 + (6\sqrt{2})i$ · (c) $5/4 - i\sqrt{6}$ · (d) $7 - 24i$ · (e) $-4 + (2\sqrt{5})i$ · (f) $(1/2)e^{i\pi/2}$

11D.9 $z^3 - 12z^2 + pz + q = 0$ where p and q are real constants. (a) Given that $z = 2 + i$ is a root of this equation, write down another root of the equation. (b) Find the value of p and the value of q. (c) Represent the roots of this equation on an Argand diagram.

11D.10 $f(z) = z^4 - z^3 + 20z^2 - 16z + 64 = (z^2 + a)(z^2 + bz + 4)$. Solve the equation $f(z) = 0$.

Exercise 11E · Loci

11E.1 Describe, in words, the locus represented by each of the following: (a) $\arg(z - 2 + 3i) = \pi/12$ · (b) $|z - 10| < |z - 6i|$ · (c) $|z + 6 - i| = 7$ · (d) $\arg(z) = 5\pi/12$

11E.2 On different Argand diagrams, sketch the following loci. Shade any regions where needed: (a) $|z - (2 + i)| = |z - (4 + 3i)|$ · (b) $\arg(z + 3i) = -\pi/3$ · (c) $|z + 4 - 5i| \leq 2$ · (d) $|z - 2i| \leq |z - 4|$

11E.3 On a single Argand diagram, sketch the loci $|z| = 4$ and $|z + 2| = |z - 4|$. Hence determine complex numbers that satisfy both loci, giving your answers in Cartesian form.

11E.4 On an Argand diagram, sketch the locus $|z| = |z + 8i|$. Find the Cartesian equation of this locus.

11E.5 Sketch the locus $|z - (3 - 6i)| = 3$ on an Argand diagram. Write down the Cartesian equation of this locus.

11E.6 Sketch the loci $\arg(z + 4 + 2i) = 5\pi/6$ and $|z - 5i| = 4$ on an Argand diagram. Determine whether or not there is a complex number, z, that satisfies both loci.

11E.7 For complex numbers z satisfying $|z - 8 - 16i| = 2\sqrt{5}$, find the least possible value of $|z|$ and the greatest possible value of $|z|$.

11E.8 By sketching the appropriate loci on an Argand diagram, find the value of z that satisfies $|z| = 13$ and $\arg(z - 12) = \pi/2$.

11E.9 On a single Argand diagram, sketch the loci $|z + 3 - 2i| = 4$ and $\arg(z + 1) = \pi/4$. Hence determine the value of z that satisfies both loci, giving your answer in Cartesian form.

11E.10 The complex number z is represented by the point P(x, y) on an Argand diagram. It is given that the locus of P is $|z - 5 - 5i| = 5$. (a) Write down the Cartesian equation of the locus of P. (b) Sketch an Argand diagram to show the locus of P. (c) Write down the greatest possible value of arg z and the least possible value of arg z.

End-of-chapter review exercise 11

R11.1 (a) Express $(5 - 2i)/(1 + 3i)$ in the form $x + iy$ where x and y are real numbers. [3]

(b) Solve $w^2 - 2w + 26 = 0$. [2]

(c) On a sketch of an Argand diagram, shade the region whose points represent complex numbers satisfying the inequality $|z + 1 - 5i| \leq 2$. [3]

R11.2 (a) The complex number z is defined as $z = k - 6i$, where k is a real value. Find and simplify expressions, in terms of k, for zz^* and z/z^*, giving your answers in the form $x + iy$ where x and y are real. [3]

(b) The complex numbers u and w are defined as $u = 4(\cos(5\pi/12) + i \sin(5\pi/12))$ and $w = 2e^{i\pi}$. Find and simplify expressions for uw and u/w, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [5]

R11.3 The complex number $w = 1 + 2i$. (a) Represent w and w^* by points P and Q on an Argand diagram with origin O and describe the polygon OPQ. [2]

(b) Given also that $u = -3 - i$, write the complex number u/w in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. [4]

R11.4 $z = 2 - 5i$. (a) Find the real values x and y such that $z^* = (2x + 1) + (4x + y)i$. [2]

(b) On an Argand diagram, show the points A, B and C representing the complex numbers z, z^* and $-z$. What type of triangle is ABC? [3]

(c) Without using a calculator, express $z^*/(-z)$: (i) in the form $x + iy$ where x and y are real [3] · (ii) in the form $r(\cos\theta + i\sin\theta)$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

R11.5 $z^2 + (4\sqrt{3})z + 13 = 0$. (a) Find the roots of this equation, giving your answers in the form $x + iy$ where x and y are real. [2]

(b) On an Argand diagram with origin O, show the position vectors OA and OB representing the roots of the equation. Describe the geometrical relationship between OA and OB. [2]

(c) Find the modulus and argument of each root. [3]

R11.6 $z = 4\sqrt{3} - 4i$. (a) Find the exact values of the modulus and argument of z. [3]

(b) Given that $w = 2\sqrt{2}(\cos(\pi/12) + i\sin(\pi/12))$, write z/w in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

R11.7 (a) Find the complex number w satisfying the equation $w^* - 2 - 2i = 3iw$. Give your answer in the form $x + iy$ where x and y are real. [5]

(b) (i) On a single Argand diagram, sketch the loci $|z - 3 - 3i| = 2$ and $\arg(z - 3 - 3i) = \pi/3$. [4]

(ii) Hence determine the value of z that satisfies both loci, giving your answer in the form x + iy, where x and y are real. [2]

R11.8 (a) $(x + iy)^2 = 7 - (6\sqrt{2})i$. Given that x and y are real numbers, find the values of x and the values of y. [5]

(b) (i) Show that $z - 3$ is a factor of $2z^3 - 4z^2 - 5z - 3$. [1]

(ii) Solve $2z^3 - 4z^2 - 5z - 3 = 0$. [4]

R11.9 (a) Given that $z_1 = 5 - 3i$ and $z_1z_2 = 21 + i$, find z_2, giving your answer in the form x + iy, where x and y are real. [2]

(b) Solve $(3z + 1)^3 = -27$. [3]

R11.10 (a) It is given that w = 1 is a root of the equation $f(w) = 2w^4 + 5w^3 - 2w^2 + w - 6 = 0$.

(i) Show that w + 3 is a factor of $2w^4 + 5w^3 - 2w^2 + w - 6$. [1]

(ii) Solve the equation. [4]

(b) (i) On an Argand diagram, sketch the locus $|z + 1 - i\sqrt{3}| = 1$. [2]

(ii) Write down the minimum value of arg z. [1]

(iii) Find the maximum value of arg z. [2]

R11.11 (a) (i) Given that $z_1 = -3/2 + (\sqrt{7}/2)i$ is a root of the equation $z^2 + pz + q = 0$, where p and q are real constants, find the value of p and the value of q. [3]

(ii) Find $|z_1|$. [1]

(b) (i) Find the roots of the equation $z^3 + 1 = 0$. [3]

(ii) On an Argand diagram, show the points A, B and C representing the roots of the equation. What type of triangle is ABC? [2]

R11.12 $z = \sqrt{5} - i$. (a) Show that $z/z^* = 2/3 - (\sqrt{5}/3)i$. [3]

(b) Find the value of $|z/z^*|$ and $\arg(z/z^*)$. [2]

(c) Find and simplify a quadratic equation with integer coefficients that has roots z/z^* and its conjugate. [2]

R11.13 The complex number z is defined by $z = (k - 4i)/(2k - i)$ where k is an integer. (a) The imaginary part of z is $\text{Im } z = 7/5$. Find the value of k. [4]

(b) Find the argument of z. [3]

R11.14 (i) Without using a calculator, solve the equation $3w + 2iw^* = 17 + 8i$, where w^* denotes the complex conjugate of w. Give your answer in the form $a + bi$. [4]

(ii) In an Argand diagram, the loci $\arg(z - 2i) = (1/6)\pi$ and $|z - 3| = |z - 3i|$ intersect at the point P. Express the complex number represented by P in the form $re^{i\theta}$, giving the exact value of θ and the value of r correct to 3 significant figures. [5]

R11.15 Throughout this question the use of a calculator is not permitted.

(i) The complex numbers u and v satisfy the equations $u + 2v = 2i$ and $iu + v = 3$. Solve the equations for u and v, giving both answers in the form $x + iy$, where x and y are real. [5]

(ii) On an Argand diagram, sketch the locus representing complex numbers z satisfying $|z + i| = 1$ and the locus representing complex numbers w satisfying $\arg(w - 2) = (3/4)\pi$. Find the least value of $|z - w|$ for points on these loci. [5]

R11.16 Throughout this question the use of a calculator is not permitted.

The complex numbers w and z satisfy the relation $w = (z + i)/(iz + 2)$.

(i) Given that $z = 1 + i$, find w, giving your answer in the form $x + iy$, where x and y are real. [4]

(ii) Given instead that $w = z$ and the real part of z is negative, find z, giving your answer in the form $x + iy$, where x and y are real. [4]

Cross-topic review exercise 4

CT4.1 Relative to the origin O, the position vectors of the points A and B are given by OA = $(-5, 0, 3)^T$ and OB = $(1, 7, 2)^T$.

(a) Find a vector equation of the line AB. [3]

(b) The line AB is perpendicular to the line L with vector equation $\mathbf{r} = (4, 2, -3)^T + \mu(m, 3, 9)^T$.

(i) Find the value of m. [3]

(ii) Show that the line AB and the line L do not intersect. [4]

CT4.2 Given that y = 0 when x = 1, solve the differential equation $xy(dy/dx) = y^2 + 4$, obtaining an expression for y^2 in terms of x. [6]

CT4.3 The point P(p, q, −1) lies on the line L with vector equation $\mathbf{r} = \mathbf{i} - \mathbf{j} + 2\mathbf{k} + \lambda(\mathbf{i} + 8\mathbf{j} + \mathbf{k})$.

(a) Find the value of each of the constants p and q. [2]

(b) The position vector of Q, relative to the origin O, is $\overrightarrow{OQ} = -10\mathbf{i} + \mathbf{j} - 5\mathbf{k}$.

(i) Find the unit vector in the direction OQ. [2]

(ii) Find angle POQ and hence find the exact area of triangle POQ. [5]

CT4.4 The complex number z is given by $z = (\sqrt{3}) + i$.

(i) Find the modulus and argument of z. [2]

(ii) The complex conjugate of z is denoted by z*. Showing your working, express in the form x + iy, where x and y are real, (a) 2z + z*, (b) iz*/z. [4]

(iii) On a sketch of an Argand diagram with origin O, show the points A and B representing the complex numbers z and iz* respectively. Prove that angle AOB = (1/6)π. [3]

CT4.5 The variables x and θ are related by the differential equation $\sin 2\theta \, (dx/d\theta) = (x + 1) \cos 2\theta$, where $0 < \theta < (1/2)\pi$. When $\theta = (1/12)\pi$, $x = 0$. Solve the differential equation, obtaining an expression for x in terms of θ, and simplifying your answer as far as possible. [7]

CT4.6 PQRS is a parallelogram. The vertices P, Q and R have position vectors relative to origin O: $OP = (1, 4, 5)^T$, $OQ = (3, 2, 0)^T$ and $OR = (2, -1, 5)^T$.

(a) Find OS. [2]

(b) Find the lengths of the sides of the parallelogram. [3]

(c) Find the interior angles of the parallelogram. [3]

CT4.7 The complex number u is defined by $u = (1 + 2i)^2/(2 + i)$.

(i) Without using a calculator and showing your working, express u in the form $x + iy$, where x and y are real. [4]

(ii) Sketch an Argand diagram showing the locus of the complex number z such that $|z - u| = |u|$. [3]

CT4.8 The complex number 2 + 2i is denoted by u.

(i) Find the modulus and argument of u. [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers 1, i and u. Shade the region whose points represent the complex numbers z which satisfy both the inequalities $|z - 1| \leq |z - i|$ and $|z - u| \leq 1$. [4]

(iii) Using your diagram, calculate the value of $|z|$ for the point in this region for which arg z is least. [3]

CT4.9 A certain substance is formed in a chemical reaction. The mass of substance formed t seconds after the start of the reaction is x grams. At any time the rate of formation of the substance is proportional to (20 − x). When t = 0, x = 0 and dx/dt = 1.

(i) Show that x and t satisfy the differential equation dx/dt = 0.05(20 − x). [2]

(ii) Find, in any form, the solution of this differential equation. [5]

(iii) Find x when t = 10, giving your answer correct to 1 decimal place. [2]

(iv) State what happens to the value of x as t becomes very large. [1]

CT4.10 The line L_1 has vector equation $\mathbf{r} = -2\mathbf{j} + \mathbf{k} + \lambda(\mathbf{i} + 3\mathbf{k})$. The line L_2 passes through the point P(−2, 1, −1) and is parallel to L_1.

(a) Write down a vector equation of the line L_2. [2]

(b) Find the shortest distance from P to the line L_1. [5]

CT4.11 In a certain country the government charges tax on each litre of petrol sold to motorists. The revenue per year is R million dollars when the rate of tax is x dollars per litre. The variation of R with x is modelled by the differential equation $dR/dx = R(1/x - 0.57)$, where R and x are taken to be continuous variables. When $x = 0.5$, $R = 16.8$.

(i) Solve the differential equation and obtain an expression for R in terms of x. [6]

(ii) This model predicts that R cannot exceed a certain amount. Find this maximum value of R. [3]

CT4.12 The complex number z is defined by $z = (9\sqrt{3} + 9i)/(\sqrt{3} - i)$. Find, showing all your working:

(i) an expression for z in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$, [5]

(ii) the two square roots of z, giving your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \leq \pi$. [3]

CT4.13 The variables x and y are related by the differential equation $dy/dx = (6ye^{3x})/(2 + e^{3x})$. Given that $y = 36$ when $x = 0$, find an expression for y in terms of x. [6]

CT4.14 (a) Show that the straight line L with vector equation $\mathbf{r} = (1, 2, -10)^T + \lambda(8, -1, 3)^T$ intersects with the line through the points A(0, 2, 7) and B(7, 1, 27) and find the position vector of this point of intersection. [5]

(b) Find the acute angle between these two lines. [4]

CT4.15 The variables x and y satisfy the differential equation $x(dy/dx) = y(1 - 2x^2)$, and it is given that $y = 2$ when $x = 1$. Solve the differential equation and obtain an expression for y in terms of x in a form not involving logarithms. [6]

CT4.16 The complex number u is defined by $u = (6 - 3i)/(1 + 2i)$.

(i) Showing all your working, find the modulus of u and show that the argument of u is $-(1/2)\pi$. [4]

(ii) For complex numbers z satisfying $\arg(z - u) = (1/4)\pi$, find the least possible value of $|z|$. [3]

(iii) For complex numbers z satisfying $|z - (1 + i)u| = 1$, find the greatest possible value of $|z|$. [3]

CT4.17 With respect to the origin O, the position vectors of the points A, B, C and D are given by $OA = (2, 1, 5)^T$, $OB = (3, m, 4)^T$, $OC = (1, -5, m + 11)^T$ and $OD = (2, -4, -2m)^T$.

(a) In the case where ABC is a right angle, find the possible values of the constant m. [3]

(b) In the case where D is the midpoint of the line BC, find the value of the constant m. [2]

(c) In the case where $m = -5$, find whether the lines AB and CD intersect. [5]

CT4.18 The complex number w is defined by $w = (22 + 4i)/(2 - i)^2$.

(i) Without using a calculator, show that $w = 2 + 4i$. [3]

(ii) It is given that p is a real number such that $(1/4)\pi \leq \arg(w + p) \leq (3/4)\pi$. Find the set of possible values of p. [3]

(iii) The complex conjugate of w is denoted by w*. The complex numbers w and w* are represented in an Argand diagram by the points S and T respectively. Find, in the form $|z - a| = k$, the equation of the circle passing through S, T and the origin. [3]

CT4.19 Given that $y = 1$ when $x = 0$, solve the differential equation $dy/dx = 4x(3y^2 + 10y + 3)$, obtaining an expression for y in terms of x. [9]

CT4.20 A tank containing water is in the form of a cone with vertex C. The axis is vertical and the semi-vertical angle is 60°. At time t = 0, the tank is full and the depth of water is H. At this instant, a tap at C is opened and water begins to flow out. The volume of water in the tank decreases at a rate proportional to $\sqrt{h}$ where h is the depth of water at time t. The tank becomes empty when t = 60.

(i) Show that h and t satisfy a differential equation of the form $dh/dt = -A\,h^{-3/2}$, where A is a positive constant. [4]

(ii) Solve the differential equation given in part (i) and obtain an expression for t in terms of h and H. [6]

(iii) Find the time at which the depth reaches (1/2)H. [1]

[The volume V of a cone of vertical height h and base radius r is given by $V = (1/3)\pi r^2 h$.]

CT4.21 (a) Find, in the form $r = a + \lambda b$, a vector equation of the line AB where the points have coordinates A(2, 5, 7) and B(9, −1, −2). [3]

(b) Find the obtuse angle between the line AB and a line in the direction of $i + 3j + 2k$. [4]

CT4.22 The complex number $3 - i$ is denoted by u. Its complex conjugate is denoted by u*.

(i) On an Argand diagram with origin O, show the points A, B and C representing the complex numbers u, u* and u* − u respectively. What type of quadrilateral is OABC? [4]

(ii) Showing your working and without using a calculator, express u*/u in the form $x + iy$, where x and y are real. [3]

(iii) By considering the argument of u*/u, prove that $\tan^{-1}(3/4) = 2\tan^{-1}(1/3)$. [3]

CT4.23 The complex number $1 + (\sqrt{2})i$ is denoted by u. The polynomial $x^4 + x^2 + 2x + 6$ is denoted by p(x).

(i) Showing your working, verify that u is a root of the equation p(x) = 0, and write down a second complex root of the equation. [4]

(ii) Find the other two roots of the equation p(x) = 0. [6]

CT4.24 A cuboid OABCDEFG has a horizontal base OABC with OA = 10 cm, width AB = 4 cm and height BF = h cm. The point M is the midpoint of CB and N is on DG such that ON = j + 3k. Unit vectors i, j and k are parallel to OA, OC and OD respectively.

(a) Write down the value of h. [1]

(b) Find the unit vector in the direction ON. [2]

(c) Find angle NME. [4]

(d) Find, in the form $r = a + \lambda b$, a vector equation of the line MF. [3]

CT4.25 The variables x and y are related by the differential equation $dy/dx = (6xe^{3x})/y^2$. It is given that $y = 2$ when $x = 0$. Solve the differential equation and hence find the value of y when $x = 0.5$, giving your answer correct to 2 decimal places. [8]

CT4.26 (i) Showing all your working and without the use of a calculator, find the square roots of the complex number $7 - (6\sqrt{2})i$. Give your answers in the form $x + iy$, where x and y are real and exact. [5]

(ii) (a) On an Argand diagram, sketch the loci of points representing complex numbers w and z such that $|w - 1 - 2i| = 1$ and $\arg(z - 1) = (3/4)\pi$. [4]

(b) Calculate the least value of $|w - z|$ for points on these loci. [2]

CT4.27 The line L_1 has vector equation $\mathbf{r} = (-2, 3, 0)^T + \lambda(1, -1, 2)^T$ and L_2 has vector equation $\mathbf{r} = (-4, 5, m)^T + \mu(-1, 0, 1)^T$.

(a) In the case where $m = 2$, show that L_1 and L_2 do not intersect. [4]

(b) Find the value of m in the case where L_1 and L_2 intersect. [2]

(c) For your value of m from part (b), find the acute angle between L_1 and L_2. [4]

CT4.28 The number of birds of a certain species in a forested region is recorded over several years. At time t years, the number of birds is N, where N is treated as a continuous variable. The variation in the number of birds is modelled by $dN/dt = (N(1800 - N))/3600$. It is given that $N = 300$ when $t = 0$.

(i) Find an expression for N in terms of t. [9]

(ii) According to the model, how many birds will there be after a long time? [1]

CT4.29 (i) Showing your working, find the square roots of the complex number $1 - (2\sqrt{6})i$. Give your answers in the form $x + iy$, where x and y are exact. [5]

(ii) On a sketch of an Argand diagram, shade the region whose points represent the complex numbers z which satisfy the inequality $|z - 3i| \leq 2$. Find the greatest value of $\arg z$ for points in this region. [4]

CT4.30 Liquid is flowing into a small tank which has a leak. Initially the tank is empty and t minutes later, the volume of liquid in the tank is V cm^3. The liquid is flowing into the tank at a constant rate of 80 cm^3 per minute. Because of the leak, liquid is being lost from the tank at a rate which, at any instant, is equal to kV cm^3 per minute where k is a positive constant.

(i) Write down a differential equation describing this situation and solve it to show that $V = (1/k)(80 - 80e^{-kt})$. [7]

(ii) It is observed that V = 500 when t = 15, so that k satisfies the equation $k = (4 - 4e^{-15k})/25$. Use an iterative formula, based on this equation, to find the value of k correct to 2 significant figures. Use an initial value of k = 0.1 and show the result of each iteration to 4 significant figures. [3]

(iii) Determine how much liquid there is in the tank 20 minutes after the liquid started flowing, and state what happens to the volume of liquid in the tank after a long time. [2]

Check yourself

CY.1 Given $z_1 = 3 - 4i$ and $z_2 = -1 + 2i$, express z_1/z_2 in the form $x + iy$. [3]

CY.2 Find the modulus and principal argument of the complex number $z = -2\sqrt{3} + 2i$, giving the argument in radians in terms of π. [3]

CY.3 The complex numbers u and w have $|u| = 6$, $\arg u = 2\pi/3$, $|w| = 2\sqrt{3}$ and $\arg w = -\pi/6$. Write uw and u/w in polar form $r(\cos\theta + i\sin\theta)$. [4]

CY.4 Find the two square roots of the complex number $-7 + 24i$, showing all algebra clearly. [5]

CY.5 Given that $3 + 2i$ is a root of the cubic equation $z^3 - 4z^2 - 3z + 52 = 0$, find the other two roots. [5]

CY.6 On an Argand diagram, sketch the region defined by $|z - 3 - 4i| \leq 2$ and $|z - 3| \geq |z - 5|$. Find the least and greatest values of $|z|$ for points in this region. **[5]**

Chapter quiz

Q1 Simplify i^{47}.

A	B	C	D
1	−1	i	−i

Q2 Simplify $(3 + 2i)(4 - 5i)$ into Cartesian form $a + bi$.

A	B	C	D
22 − 7i	2 − 7i	22 + 7i	2 + 7i

Q3 Find the quotient $(2 + 3i)/(1 - i)$ in the form $a + bi$.

A	B	C	D
−1/2 + 5i/2	−1 + 5i	5/2 + 5i/2	5 − i

Q4 If $(x + 2y) + i(2x - y) = 7 + 4i$ where x, y are real, find x.

A	B	C	D
x = 3	x = 2	x = 4	x = 1

Q5 Find the modulus of the complex number $z = -5 + 12i$.

A	B	C	D
7	13	17	√119

Q6 Find the principal argument arg(z) of $z = -1 - i\sqrt{3}$ in radians.

A	B	C	D
$-2\pi/3$	$2\pi/3$	$-\pi/3$	$4\pi/3$

Q7 Given $|u| = 6$, $\arg(u) = \pi/4$ and $|w| = 2$, $\arg(w) = 3\pi/4$, find $\arg(u/w)$.

A	B	C	D
$-\pi/2$	$\pi/2$	π	$-\pi$

Q8 Find the two square roots of $-15 + 8i$.

A	B	C	D
$\pm(1 + 4i)$	$\pm(1 - 4i)$	$\pm(4 + i)$	$\pm(2 + 3i)$

Q9 Given that $2 - 3i$ is a root of $z^3 - 3z^2 + 9z + 13 = 0$, name the other roots.

A	B	C	D
$2 + 3i$ and -1	$2 + 3i$ and 1	$-2 + 3i$ and -1	$-2 - 3i$ and 1

Q10 Describe the locus given by $|z - 3 + 4i| = 5$.

A	B	C	D
Circle centre $(3, -4)$, radius 5	Circle centre $(-3, 4)$, radius 5	Circle centre $(3, 4)$, radius 25	Straight line through $(3, -4)$

Q11 Describe the geometric locus given by $|z - 4| = |z - 2i|$.

A	B	C	D
Perpendicular bisector of line joining $(4, 0)$ and $(0, 2)$	Perpendicular bisector of line joining $(-4, 0)$ and $(0, -2)$	Circle with diameter connecting $(4, 0)$ and $(0, 2)$	Straight line through $(4, 0)$ and $(0, 2)$

Q12 What does the equation $\arg(z - 2 - i) = \pi/3$ represent in the Argand plane?

A	B	C	D
A half-line starting from $(2, 1)$ at angle $\pi/3$ to the real axis	A full straight line passing through $(2, 1)$ with gradient $\sqrt{3}$	A circle of radius $\pi/3$ centred at $(2, 1)$	A line segment from the origin $(0, 0)$ to $(2, 1)$

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p23-11-complex-numbers/