P23·10

Differential equations

CHAPTER ASSIGNMENT

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. State constants of integration, give exact forms where required, and interpret rates and limits clearly in context.

Practice questions based on *Cambridge International AS & A Level Mathematics: Pure Mathematics 2 & 3 Coursebook* (Cambridge University Press, Sue Pemberton and Julianne Hughes), Chapter 10 (Differential equations), Exercises 10A, 10B and End-of-chapter review exercise 10. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 10A · Technique of separating variables

10A.1 Find the general solution of each of the following differential equations:

(a) $dy/dx = x^3 - 6$

(b) $dA/dr = 2\pi r$

(c) $s^2 (ds/dt) = \cos(t + 5)$

(d) $V^{-1} (dV/dt) = 2$

(e) $dy/dx = (y + 1)/x$

(f) $x + \sec x \sin y \left(\frac{dy}{dx}\right) = 1$

10A.2 Find the particular solution for each of the following differential equations, using the values given:

(a) $\cos^2 x \left(\frac{dy}{dx}\right) = y$, given $y = 5$ when $x = 0$

(b) $xy\,(dy/dx) = (x^3 - x)/(1 - \sqrt{y})$, given $y = 4$ when $x = 3$

10A.3 Given that $dy/dx = e^{x - y}$ and $y = \ln 2$ when $x = 0$, obtain an expression for y in terms of x.

10A.4 The gradient of a curve at the point (x, y) is $5y^3x$.

(a) Write down the differential equation satisfied by y.

(b) Given that the curve passes through the point $(1, 1)$, find y^2 as a function of x.

10A.5 (a) Express $1/((2 - x)(x + 1))$ in the form $A/(2 - x) + B/(x + 1)$.

(b) The variables x and t satisfy the differential equation $dx/dt = (2 - x)(x + 1)$. It is given that $t = 0$ when $x = 0$.

(i) Solve this differential equation and obtain an expression for x in terms of t.

(ii) State what happens to the value of x when t becomes large.

10A.6 The variables w and t satisfy the differential equation $dw/dt = 0.001(100 - w)^2$. Given that when $t = 0$, $w = 0$, show that the solution to the differential equation can be written in the form $w = 100 - 1000/(t + 10)$.

10A.7 Solve the differential equation $4\,(dx/dt) = (3x^2 - x)\cos t$ given that $x = 0.4$ when $t = 0$. Give your answer as an expression for x in terms of t.

10A.8 Find the general solution of the differential equation $xy\,(dy/dx) = (x^2 - 1)/e^{y+1}$.

10A.9 Find the equation of the curve that satisfies the equation $dy/dx = (1 + y^2)/(4 + x^2)$ and passes through the point (0, 1). Give your answer in the form $y = f(x)$.

10A.10 Find the general solution of the equation $x^3\,(dy/dx) + 2y = 1$.

10A.11 Solve the differential equation $dx/dt = 4x \cos^2 t$ given that $x = 1$ when $t = 0$. Give your answer as an expression for x in terms of t.

10A.12 (a) Use the substitution $u = t^2$ to find $\int t\, e^{t^2}\, dt$.

(b) Solve the differential equation $dx/dt = t(e^{t^2} + 5)/x^2$ given that $x = 1$ when $t = 0$.

10A.13 At a point with coordinates (x, y) the gradient of a curve is directly proportional to xy^2. At point P(1, 3) the gradient is 6. Find and simplify an expression for y in terms of x.

10A.14 The variables x and t satisfy $(3x + 2x^3)(dx/dt) = k(3x^2 + x^4)$ for $x > 0$ where k is a constant. When $t = 0$, $x = 1$ and when $t = 0.5$, $x = 2$. Solve the differential equation, finding the exact value of k, and show that $7^{2t} = (3x^2 + x^4)/4$.

10A.15 (a) Use the substitution $u = \ln x$ to find $\int 1/(x \ln x)\,dx$.

(b) Given that $x > 0$, $y > 0$, find the general solution of $\ln x \,(dy/dx) = y/x$.

10A.16 The amount of light-sensitive medicine m (in milligrams) remaining in treated skin t hours after treatment satisfies $dm/dt = -0.05m$. Initially $m = 65$ mg. Safe threshold for bright light is $m \le 6$ mg. How long must the patient wait before going out into bright light?

Exercise 10B · Forming differential equations from problems

10B.1 Write each of the following as a differential equation (do not solve):

(a) Height h of water in a tank decreases at a rate proportional to h^2.

(b) Number of infected cells n in a body increases at a rate proportional to n.

(c) Velocity v of a particle in fluid decreases at a rate proportional to $v(v + 1)$.

(d) Volume V of oil leaking from an engine decreases at a rate proportional to V.

(e) Cost C of an item increases at a rate proportional to C^3.

(f) Speed v of a falling body changes at a rate inversely proportional to $e^{v/2}$.

10B.2 Water depth x metres in a leaking tank t seconds after leaking begins decreases at a rate proportional to $\sqrt{x}$.

(a) Write down a differential equation relating x, t and a constant of proportionality.

(b) Initially x = 6.25 m. When t = 120 s, x = 4 m. Solve the differential equation, expressing t in terms of x.

(c) Find the time it takes for the depth of water to halve from its initial value.

10B.3 The cost A ($/kg) of designing an aircraft t years after 1950 increases at a rate proportional to A.

(a) Write down a differential equation satisfied by A.

(b) In 1950 (t = 0), A = 12. Show that $A = 12\,e^{kt}$, where k is a positive constant.

(c) When t = 25 years, A = $48. Find the cost after 60 years.

10B.4 Liquid temperature x (°C) after t seconds increases at a rate proportional to (100 − x). Initially x = 25 °C.

(a) Form a differential equation for dx/dt with proportionality constant k.

(b) Solve the differential equation to obtain x in terms of t and k.

(c) When t = 180 s, x = 85 °C. Find the temperature after 195 seconds.

(d) State the maximum temperature the liquid cannot exceed according to the model.

10B.5 An inverted circular cone of base radius 10 cm and height 15 cm contains liquid paint leaking from the vertex at 16 cm^3/s. When the paint depth is h cm:

(a) Show that the volume satisfies $dV/dh = (4/9)\pi h^2$.

(b) Find an expression for dh/dt.

(c) Solve the differential equation for t in terms of h.

(d) Find the time required for depth to fall from 15 cm to 7.5 cm.

10B.6 Population size P at time t minutes increases at a rate proportional to P.

(a) Write down the differential equation satisfied by P.

(b) Initial population is P_0. Show that $P = P_0 e^{kt}$ with $k > 0$.

(c) When $t = 2$ min, $P = 1.5 P_0$. Find when P reaches $3 P_0$.

10B.7 A spherical balloon radius r cm increases at a rate inversely proportional to √r. When t = 4 s, dr/dt = 1.4 cm/s and r = 7.84 cm.

(a) Form a differential equation relating r and t.

(b) Solve to find r in terms of t.

(c) Find the initial volume of air in the balloon at t = 0.

10B.8 Hot food at 94 °C cools in cold water held at 7 °C. After 10 minutes the food temperature is 54 °C. The cooling rate is proportional to the temperature difference between food and water. Find the time at which the food cools to room temperature (18 °C).

10B.9 Carbon-14 mass m decays at a rate proportional to m, with half-life 5700 years. If the initial mass is m_0, determine the fraction of carbon-14 remaining after 2500 years.

10B.10 Hair length L grows at a constant rate. Initially L = 20 cm, and after 20 weeks L = 26 cm. Form and solve a differential equation relating L and time t in weeks.

10B.11 Water at 4 °C warms in air at 24 °C at a rate proportional to (24 − θ). After 2 minutes its temperature is 11 °C.

(a) How long does it take for the water to reach 20 °C?

(b) What temperature does the water eventually reach as $t \to \infty$?

10B.12 Customer count n at a shop t months after opening increases at a rate proportional to $\sqrt{n}$.

(a) Form and solve a differential equation relating n and t.

(b) Given n = 0 at t = 0 and n = 3600 at t = 6, find the number of complete months for n to reach 6800.

(c) For capacity 14 000 customers/month, show that capacity is reached during the 12th month.

10B.13 Bacterial count in culture grows at a rate proportional to the population present. Initially 5000 bacteria are present, growing to 1 000 000 after 3.5 hours.

(a) Find the doubling time of the bacteria.

(b) Discuss the assumption of unlimited exponential growth and its limitations.

End-of-chapter review exercise 10

R10.1 Variables x and y satisfy $dy/dx = x^2 e^{y + 2x}$ with $y = 0$ when $x = 0$. Solve to express y in terms of x.

R10.2 Bird population x in a region satisfies $dx/dt = x(2500 - x)/5000$ with $x = 500$ when $t = 0$.

(a) Find an expression for x in terms of t.

(b) State the limiting bird population in the long term.

R10.3 Solve $y^3 (dy/dx) = 1 + y^4$ given $y = 2$ when $x = 0$, obtaining an expression for y^4 in terms of x.

R10.4 The gradient at (x, y) is proportional to $x\sqrt{y}$, and at $(3, 4)$ the gradient is -5.

(a) Form and solve a differential equation to find the equation of the curve.

(b) Find the gradient at $(-3, 4)$.

R10.5 (a) Given $x < 5$, evaluate $\int 50/((5 - x)(10 - x))\,dx$.

(b) Reactants A ($5 - x$ g) and B ($10 - x$ g) produce product C (x g) with reaction rate proportional to $(5 - x)(10 - x)$. Initially $x = 0$ and $dx/dt = 1$ g/h.

(i) Show that $50\,(dx/dt) = (5 - x)(10 - x)$.

(ii) Solve the equation to express x in terms of t.

(iii) State the limiting mass of chemical C produced as $t \to \infty$.

R10.6 Variables x and y satisfy $dy/dx = (x\,e^x)/(5y^4)$ with $y = 4$ when $x = 0$. Find the particular solution and compute y when $x = 3.5$.

R10.7 (a) Use $u = \sqrt{(y^4 - 1)}$ to find $\int y^3/\sqrt{(y^4 - 1)}\, dy$.

(b) Given $dy/dx = ((2x + 1)\sqrt{(y^4 - 1)})/(x\, y^3)$ and $y = \sqrt{3}$ when $x = 1$, find a relationship between x and y.

R10.8 Height h metres of a tree t years after planting satisfies dh/dt proportional to $8 - h$. Initially $h = 0.5$ m, and when $t = 5$, $h = 2$ m.

(a) Form and solve a differential equation to obtain h in terms of t.

(b) What will the tree height approach as $t \to \infty$?

R10.9 Solve $dy/dx = e^{3(x + y)}$ with $y = 0$ when $x = 0$, expressing y in terms of x.

R10.10 Substance mass x decreases at a rate proportional to xt with k > 0.

(a) Form the differential equation for dx/dt.

(b) Solve to find a relation between x, t and k.

(c) Initially x = 150 g and at t = 10 s, x = 120 g. Find the time when x = 1 g.

R10.11 (a) Use partial fractions to find $\int 1/(P(5 - P))\, dP$.

(b) Given P = 3 when t = 0, solve $dP/dt = P(5 - P)$ to express P in terms of t.

(c) Describe the limiting behavior of P as $t \to \infty$.

R10.12 Population P (in millions) satisfies $dP/dt = k\,P(100 - P)/100$. Given P = 50 in 2000 and P = 60 in 2010, predict the population in 2025.

R10.13 (a) Use substitution $u = x^2$ to find $\int x \sin(x^2)\,dx$.

(b) Given $dy/dx = (x \sin(x^2))/y$ with $y = -1$ when $x = 0$, find a relationship between x and y.

R10.14 Solve $dy/dx = \tan x / e^{3y}$ given $y = 0$ when $x = 0$, in the form $y = f(x)$.

R10.15 In a population of 2000 animals, infection count x increases at a rate proportional to $x(2000 - x)$. Initially $x = 500$ and $dx/dt = 50$ per hour.

(a) Form and solve a differential equation to find t in terms of x.

(b) Calculate the time required for 1900 animals to be infected.

R10.16 Variables x and y satisfy $x(dy/dx) = 1 - y^2$ with $y = 0$ when $x = 2$. Solve to find an expression for y in terms of x.

R10.17 Variables x and θ satisfy $dx/d\theta = (x + 2)\sin^2 2\theta$ with $x = 0$ when $\theta = 0$. Solve the differential equation and calculate x when $\theta = \pi/4$ to 3 significant figures.

Check yourself

C10.1 Find the particular solution of $dy/dx = 2x(y + 3)$ given that $y = 1$ when $x = 0$.

C10.2 Solve the differential equation $dy/dx = (x^2 + 1)\, e^{-2y}$ given that $y = 0$ when $x = 0$.

C10.3 A heated metal bar cools in a room at 22 °C according to Newton's law of cooling. Its temperature drops from 90 °C to 56 °C in 15 minutes. How many minutes from the start does it take to reach 30 °C?

C10.4 Solve $dx/dt = 3x(10 - x)$ using partial fractions, given that $x = 2$ when $t = 0$. Express x in terms of t.

C10.5 The radius r cm of a circular ripple grows at a rate inversely proportional to r + 1. Initially r = 1 cm, and after 3 seconds r = 3 cm. Find r after 8 seconds.

Chapter quiz

Q1 Find the general solution of dy/dx = 2xy for y > 0.

A	B	C	D
$y = A\,e^{x^2}$	$y = x^2 + C$	$y = A\,e^{2x}$	$y^2 = x^2 + C$

Q2 Find the general solution of dy/dx = (x + 1)/y.

A	B	C	D
$y^2 = x^2 + 2x + C$	$y^2 = 2x^2 + 2x + C$	$y = x^2 + 2x + C$	$y^2 = (x + 1)^2 + C$

Q3 Given $dy/dx = 3x^2y$ and y = 2 when x = 0, find y in terms of x.

A	B	C	D
$y = 2\,e^{x^3}$	$y = e^{x^3} + 1$	$y = 2\,e^{3x^2}$	$y = x^3 + 2$

Q4 Solve dy/dx = y cos x given y = 1 when x = 0.

A	B	C	D
$y = e^{\sin x}$	$y = \sin x + 1$	$y = e^{-\sin x}$	$y = e^{\cos x - 1}$

Q5 Which differential equation models "the rate of increase of P is proportional to P"?

A	B	C	D
dP/dt = kP	dP/dt = k/P	dP/dt = k + P	dP/dt = P/k

Q6 The rate of decrease of volume V is inversely proportional to V. Which equation is correct?

A	B	C	D
$dV/dt = -k/V$	$dV/dt = -kV$	$dV/dt = k/V$	$dV/dt = -k\,V^2$

Q7 A hot drink cools in a room at 20 °C. Newton's law of cooling gives which equation for θ?

A	B	C	D
$d\theta/dt = -k(\theta - 20)$	$d\theta/dt = -k(20 - \theta)$	$d\theta/dt = -k\theta + 20$	$d\theta/dt = -20k\,\theta$

Q8 Separating $dx/dt = x(4 - x)$ leads to which integral on the left?

A	B	C	D
$\int [1/(x(4 - x))]\,dx$	$\int x(4 - x)\,dx$	$\int [(4 - x)/x]\,dx$	$\int [x/(4 - x)]\,dx$

Q9 If $dx/dt = 0.02x(500 - x)$ for $x > 0$, what does x approach as $t \to \infty$?

A	B	C	D
500	0	250	Infinity

Q10 If $dm/dt = -km$ and mass halves every 10 years, what is the exact value of k?

A	B	C	D
$(\ln 2)/10$	$(\ln 10)/2$	$10 \ln 2$	e^{-10}

Q11 Solve $dy/dx = 2x\,y^2$ given $y = 1$ when $x = 0$.

A	B	C	D
$y = 1/(1 - x^2)$	$y = 1/(1 + x^2)$	$y = x^2 + 1$	$y = 1/(x^2 - 1)$

Q12 Solve $dy/dx = (x + 1)\,e^{-y}$ given $y = 0$ when $x = 0$.

A	B	C	D
$y = \ln(x^2/2 + x + 1)$	$y = e^{x^2/2 + x}$	$y = x^2/2 + x$	$y = \ln(x^2 + 2x + 1)$

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p23-10-differential-equations/