P1·6

Integration

CHAPTER ASSIGNMENT

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. Give exact answers (fractions, surds, π) unless a question asks for decimal places or significant figures. ∫(a to b) f(x) dx means the definite integral of f(x) from x = a to x = b, and ∫ f(x) dx with no limits is an indefinite integral. Fractions are written a/b, with brackets where needed.

Practice questions from *Pure Mathematics 1* (Hodder Education, Sophie Goldie), Chapter 6 (Integration), Exercises 6A to 6H, the chapter's activities and its "Think about it" questions, for classroom use. Where a question needs a diagram, the diagram is described in words. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 6A · Reversing differentiation

6A.1 Given that $dy/dx = 6x^2 + 5$: (i) find the general solution of the differential equation · (ii) find the equation of the curve with gradient function dy/dx which passes through (1, 9) · (iii) hence show that (−1, −5) also lies on the curve.

6A.2 The gradient function of a curve is dy/dx = 4x and the curve passes through the point (1, 5). (i) Find the equation of the curve. (ii) Find the value of y when x = −1.

6A.3 The curve C passes through the point (2, 10) and its gradient at any point is given by $dy/dx = 6x^2$. (i) Find the equation of the curve C. (ii) Show that the point (1, −4) lies on the curve.

6A.4 A stone is thrown upwards out of a window, and the rate of change of its height (h metres) is given by $dh/dt = 15 - 10t$ where t is the time (in seconds). When $t = 0$, $h = 20$. (i) Show that the solution of the differential equation, under the given conditions, is $h = 20 + 15t - 5t^2$. (ii) For what value of t does $h = 0$? (Assume $t \geq 0$.)

6A.5 (i) Find the general solution of the differential equation $dy/dx = 5$. (ii) Find the particular solution which passes through the point (1, 8). (iii) Sketch the graph of this particular solution.

6A.6 The gradient function of a curve is $3x^2 - 3$. The curve has two stationary points. One is a maximum with a y value of 5 and the other is a minimum with a y value of 1. (i) Find the value of x at each stationary point. Make it clear in your solution how you know which corresponds to the maximum and which to the minimum. (ii) Use the gradient function and one of your points from part (i) to find the equation of the curve. (iii) Sketch the curve.

6A.7 A curve passes through the point (4, 1) and its gradient at any point is given by $dy/dx = 2x - 6$. (i) Find the equation of the curve. (ii) Draw a sketch of the curve and state whether it passes under, over or through the point (1, 4).

6A.8 A curve passes through the point (2, 3). The gradient of the curve is given by $dy/dx = 3x^2 - 2x - 1$. (i) Find y in terms of x. (ii) Find the co-ordinates of any stationary points of the graph of y. (iii) Sketch the graph of y against x, marking the co-ordinates of any stationary points and the point where the curve cuts the y axis. [MEI]

6A.9 The gradient of a curve is given by $dy/dx = 3x^2 - 8x + 5$. The curve passes through the point (0, 3). (i) Find the equation of the curve. (ii) Find the co-ordinates of the two stationary points on the curve. State, with a reason, the nature of each stationary point. (iii) State the range of values of k for which the curve has three distinct intersections with the line $y = k$. (iv) State the range of values of x for which the curve has a negative gradient. Find the x co-ordinate of the point within this range where the curve is steepest. [MEI]

6A.10 A curve is such that $dy/dx = \sqrt{x}$. Given that the point (9, 20) lies on the curve, find the equation of the curve.

6A.11 A curve is such that $dy/dx = 2/x^2 - 3$. Given that the point (2, 10) lies on the curve, find the equation of the curve.

6A.12 A curve is such that $dy/dx = \sqrt{x} + 1/x^2$. Given that the point (1, 5) lies on the curve, find the equation of the curve.

6A.13 A curve is such that $dy/dx = 3x^2 + 5$. Given that the point (1, 8) lies on the curve, find the equation of the curve.

6A.14 A curve is such that $dy/dx = 3\sqrt{x} - 9$ and the point (4, 0) lies on the curve. (i) Find the equation of the curve. (ii) Find the x co-ordinate of the stationary point on the curve and determine the nature of the stationary point.

6A.15 The equation of a curve is such that $dy/dx = 3/\sqrt{x} - x$. Given that the curve passes through the point (4, 6), find the equation of the curve. [Cambridge AS & A Level Mathematics 9709, Paper 12 Q1 November 2009]

6A.16 A curve is such that $dy/dx = 4 - x$ and the point P(2, 9) lies on the curve. The normal to the curve at P meets the curve again at Q. Find (i) the equation of the curve, (ii) the equation of the normal to the curve at P, (iii) the co-ordinates of Q. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 November 2007]

Exercise 6B · Definite and indefinite integrals, areas under curves

6B.1 Find the following indefinite integrals. (i) $\int 3x^2\,dx$ · (ii) $\int (5x^4 + 7x^6)\,dx$ · (iii) $\int (6x^2 + 5)\,dx$ · (iv) $\int (x^3 + x^2 + x + 1)\,dx$ · (v) $\int (11x^{10} + 10x^9)\,dx$ · (vi) $\int (3x^2 + 2x + 1)\,dx$ · (vii) $\int (x^2 + 5)\,dx$ · (viii) $\int 5\,dx$ · (ix) $\int (6x^2 + 4x)\,dx$ · (x) $\int (x^4 + 3x^2 + 2x + 1)\,dx$

6B.2 Find the following indefinite integrals. (i) $\int 10x^{-4}\,dx$ · (ii) $\int (2x - 3x^{-4})\,dx$ · (iii) $\int (2 + x^3 + 5x^{-3})\,dx$ · (iv) $\int (6x^2 - 7x^{-2})\,dx$ · (v) $\int 5x^{1/4}\,dx$ · (vi) $\int 1/x^4\,dx$ · (vii) $\int \sqrt{x}\,dx$ · (viii) $\int (2x^4 - 4/x^2)\,dx$

6B.3 Evaluate the following definite integrals. (i) $\int_1^2 2x\,dx$ · (ii) $\int_0^3 2x\,dx$ · (iii) $\int_0^3 3x^2\,dx$ · (iv) $\int_1^5 x\,dx$ · (v) $\int_5^6 (2x + 1)\,dx$ · (vi) $\int_{-1}^2 (2x + 4)\,dx$ · (vii) $\int_3^5 (3x^2 + 2x)\,dx$ · (viii) $\int_0^1 x^5\,dx$ · (ix) $\int_{-2}^{-1} (x^4 + x^3)\,dx$ · (x) $\int_{-1}^1 x^3\,dx$ · (xi) $\int_{-5}^4 (x^3 + 3x)\,dx$ · (xii) $\int_{-3}^{-2} 5\,dx$

6B.4 Evaluate the following definite integrals. (i) $\int_1^4 3x^{-2}\,dx$ · (ii) $\int_2^4 8x^{-3}\,dx$ · (iii) $\int_1^4 12x^{1/2}\,dx$ · (iv) $\int_{-3}^{-1} 6/x^3\,dx$ · (v) $\int_{0.5}^{2} (x^2 + 3x + 4)/x^4\,dx$ · (vi) $\int_4^9 (\sqrt{x} - 1/\sqrt{x})\,dx$

6B.5 The graph of $y = 2x$ is shown (a straight line through the origin). The shaded region is bounded by $y = 2x$, the x axis and the lines $x = 2$ and $x = 3$; A and B are the points on the line where $x = 2$ and $x = 3$. (i) Find the co-ordinates of the points A and B in the diagram. (ii) Use the formula for the area of a trapezium to find the area of the shaded region. (iii) Find the area of the shaded region as $\int(2 \text{ to } 3)\ 2x\ dx$, and confirm that your answer is the same as that for part (ii). (iv) The method of part (ii) cannot be used to find the area under the curve $y = x^2$ bounded by the lines $x = 2$ and $x = 3$. Why?

6B.6 (i) Sketch the curve $y = x^2$ for $-1 \leq x \leq 3$ and shade the area bounded by the curve, the lines $x = 1$ and $x = 2$ and the x axis. (ii) Find, by integration, the area of the region you have shaded.

6B.7 (i) Sketch the curve $y = 4 - x^2$ for $-3 \leq x \leq 3$. (ii) For what values of x is the curve above the x axis? (iii) Find the area between the curve and the x axis when the curve is above the x axis.

6B.8 (i) Sketch the graph of $y = (x - 2)^2$ for values of x between $x = -1$ and $x = +5$. Shade the area under the curve, between $x = 0$ and $x = 2$. (ii) Calculate the area you have shaded. [MEI]

6B.9 The graph of $y = \sqrt{x} + 1/\sqrt{x}$ for $x > 0$ is drawn (it falls steeply from a very large value near the y axis to a minimum, then rises slowly). The shaded region is bounded by the curve, the x axis and the lines $x = 1$ and $x = 9$. Find its area.

6B.10 (i) Sketch the graph of $y = (x + 1)^2$ for values of x between $x = -1$ and $x = 4$. (ii) Shade the area under the curve between $x = 1$, $x = 3$ and the x axis. Calculate this area. [MEI]

6B.11 (i) Sketch the curves $y = x^2$ and $y = x^3$ for $0 \leq x \leq 2$. (ii) Which is the higher curve within the region $0 < x < 1$? (iii) Find the area under each curve for $0 \leq x \leq 1$. (iv) Which would you expect to be greater, $\int (1 \text{ to } 2)\ x^2\ dx$ or $\int (1 \text{ to } 2)\ x^3\ dx$? Explain your answer in terms of your sketches, and confirm it by calculation.

6B.12 (i) Sketch the curve $y = x^2 - 1$ for $-3 \leq x \leq 3$. (ii) Find the area of the region bounded by $y = x^2 - 1$, the line $x = 2$ and the x axis. (iii) Sketch the curve $y = x^2 - 2x$ for $-2 \leq x \leq 4$. (iv) Find the area of the region bounded by $y = x^2 - 2x$, the line $x = 3$ and the x axis. (v) Comment on your answers to parts (ii) and (iv).

6B.13 (i) Shade, on a suitable sketch, the region with an area given by $\int(-1 \text{ to } 2)\ (9 - x^2)\ dx$. (ii) Find the area of the shaded region.

6B.14 (i) Sketch the curve with equation $y = x^2 + 1$ for $-3 \leq x \leq 3$. (ii) Find the area of the region bounded by the curve, the lines $x = 2$ and $x = 3$, and the x axis. (iii) Predict, with reasons, the value of $\int(-3 \text{ to } -2)\ (x^2 + 1)\ dx$. (iv) Evaluate $\int(-3 \text{ to } -2)\ (x^2 + 1)\ dx$.

6B.15 (i) Sketch the curve with equation $y = x^2 - 2x + 1$ for $-1 \leq x \leq 4$. (ii) State, with reasons, which area you would expect from your sketch to be larger: $\int(-1 \text{ to } 3)\ (x^2 - 2x + 1)\ dx$ or $\int(0 \text{ to } 4)\ (x^2 - 2x + 1)\ dx$. (iii) Calculate the values of the two integrals. Was your answer to part (ii) correct?

6B.16 (i) Sketch the curve with equation $y = x^3 - 6x^2 + 11x - 6$ for $0 \leq x \leq 4$. (ii) Shade the regions with areas given by (a) $\int$(1 to 2) $(x^3 - 6x^2 + 11x - 6)\,dx$ and (b) $\int$(3 to 4) $(x^3 - 6x^2 + 11x - 6)\,dx$. (iii) Find the values of these two areas. (iv) Find the value of $\int$(1 to 1.5) $(x^3 - 6x^2 + 11x - 6)\,dx$. What does this, taken together with one of your answers to part (iii), indicate to you about the position of the maximum point between $x = 1$ and $x = 2$?

6B.17 Find the area of the region enclosed by the curve $y = 3\sqrt{x}$, the x axis and the lines $x = 0$ and $x = 4$.

6B.18 A curve has equation $y = 4/\sqrt{x}$. (i) The normal to the curve at the point (4, 2) meets the x axis at P and the y axis at Q. Find the length of PQ, correct to 3 significant figures. (ii) Find the area of the region enclosed by the curve, the x axis and the lines $x = 1$ and $x = 4$. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2005]

6B.19 The diagram shows a curve (it falls from the point (1, 18) to the point (4, 3)) for which $dy/dx = -k/x^3$, where k is a constant. The curve passes through the points (1, 18) and (4, 3). The point P lies on the curve and has x co-ordinate 1.6; the region shaded is under the curve between $x = 1$ and $x = 1.6$, down to the x axis. (i) Show, by integration, that the equation of the curve is $y = 16/x^2 + 2$. (ii) Find the area of the shaded region. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2008]

6B.20 A curve is such that $dy/dx = 16/x^3$, and (1, 4) is a point on the curve. (i) Find the equation of the curve. (ii) A line with gradient −1/2 is a normal to the curve. Find the equation of this normal, giving your answer in the form ax + by = c. (iii) Find the area of the region enclosed by the curve, the x axis and the lines x = 1 and x = 2. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 November 2005]

6B.21 The equation of a curve is $y = 2x + 8/x^2$. (i) Obtain expressions for dy/dx and d^2y/dx^2. (ii) Find the co-ordinates of the stationary point on the curve and determine the nature of the stationary point. (iii) Show that the normal to the curve at the point (−2, −2) intersects the x axis at the point (−10, 0). (iv) Find the area of the region enclosed by the curve, the x axis and the lines x = 1 and x = 2. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 June 2007]

Activities in the text

6.1 Activity 6.1: The area under $y = x^2 + 1$ between $x = 1$ and $x = 5$ can be estimated by rectangles of equal width. With 4 rectangles the area A satisfies $34 \leq A \leq 58$. Use ICT (a spreadsheet or graphing software) to calculate the lower and upper estimates for 8, 16, 32 and 64 rectangles and use them to get the bounds closer together.

6.2 Activity 6.2: The region bounded by the graph of $y = x + 3$, the x axis and the lines $x = a$ and $x = b$ is shaded. (i) Find the shaded area, A, by considering it as the difference between the two trapezia (one with its right edge at $x = b$, the other with its right edge at $x = a$, each starting at the y axis). (ii) Show that the expression for A you obtained in part (i) may be written as $[x^2/2 + 3x]$ between $x = a$ and $x = b$. (iii) Show that you obtain the same answer for A by integration.

6.3 Activity 6.3: (i) Use the chain rule to differentiate these. (a) $(x - 2)^4$ · (b) $(2x + 5)^7$ · (c) $1/(2x - 1)^3$ · (d) $\sqrt{(1 - 8x)}$. (ii) Use your answers to part (i) to find these. (a) $\int 4(x - 2)^3\,dx$ · (b) $\int (x - 2)^3\,dx$ · (c) $\int 7(2x + 5)^6\,dx$ · (d) $\int 28(2x + 5)^6\,dx$ · (e) $\int 6(2x - 1)^{-4}\,dx$ · (f) $\int 1/(2x - 1)^4\,dx$ · (g) $\int -4/\sqrt{(1 - 8x)}\,dx$ · (h) $\int 8/\sqrt{(1 - 8x)}\,dx$

6.4 Activity 6.4: The graph of $y = 1/x^2$ is drawn (it falls steeply near the y axis and flattens out towards the x axis); the shaded region is given by $\int (1 \text{ to } \infty)\ 1/x^2\,dx$. (i) Work out the value of $\int (1 \text{ to } b)\ 1/x^2\,dx$ when (a) $b = 2$ · (b) $b = 3$ · (c) $b = 10$ · (d) $b = 100$ · (e) $b = 10\,000$. (ii) What do you think the value of $\int (1 \text{ to } \infty)\ 1/x^2\,dx$ is?

6.5 Think about it: What is the value of $\int_{a}^{\infty} 1/x^2\,dx$? What can you say about $\int_{0}^{\infty} 1/x^2\,dx$?

6.6 Think about it: (1) Describe the solid of revolution obtained by a rotation through 360° of (i) a rectangle about one side, (ii) a semi-circle about its diameter, (iii) a circle about a line outside the circle. (2) Calculate the volume of the solid obtained when the region between $y = x$, the x axis and the lines $x = 1$ and $x = 2$ is rotated through 360° about the x axis (a cone with its tip cut off), leaving your answer as a multiple of π.

6.7 Think about it: How would you prove that the volume of revolution about the y axis is $V = \int_{p}^{q} \pi x^2\,dy$?

Exercise 6C · Areas below the x axis

6C.1 Sketch each of these curves and find the area between the curve and the x axis between the given bounds. (i) $y = x^3$ between $x = -3$ and $x = 0$. (ii) $y = x^2 - 4$ between $x = -1$ and $x = 2$. (iii) $y = x^5 - 2$ between $x = -1$ and $x = 0$. (iv) $y = 3x^2 - 4x$ between $x = 0$ and $x = 1$. (v) $y = x^4 - x^2$ between $x = -1$ and $x = 1$. (vi) $y = 4x^3 - 3x^2$ between $x = -1$ and $x = 0.5$. (vii) $y = x^5 - x^3$ between $x = -1$ and $x = 1$. (viii) $y = x^2 - x - 2$ between $x = -2$ and $x = 3$. (ix) $y = x^3 + x^2 - 2x$ between $x = -3$ and $x = 2$. (x) $y = x^3 + x^2$ between $x = -2$ and $x = 2$.

6C.2 The diagram shows a sketch of part of the curve with equation $y = 5x^4 - x^5$ (it starts at the origin, rises to a maximum and falls through the x axis); the region enclosed by the curve and the x axis is shaded. (i) Find dy/dx. Calculate the co-ordinates of the stationary points. (ii) Calculate the area of the shaded region enclosed by the curve and the x axis. (iii) Evaluate $\int_0^6 x^4(5 - x)\,dx$ and comment on your result. [MEI]

6C.3 (i) (a) Find $\int_{1/4}^{1/2} (1/x^3 - 8)\,dx$. (b) Find $\int_{1/2}^{1} (1/x^3 - 8)\,dx$. (ii) Hence find the total area of the regions bounded by the curve $y = 1/x^3 - 8$, the lines $x = 1/4$ and $x = 1$ and the x axis.

6C.4 (i) (a) Find $\int(0 \text{ to } 4)\ 2x(\sqrt{x} - 2)\ dx$. (b) Find $\int(4 \text{ to } 9)\ 2x(\sqrt{x} - 2)\ dx$. (ii) Hence find the total area of the regions bounded by the curve $y = 2x(\sqrt{x} - 2)$, the line $x = 9$ and the x axis.

Exercise 6D · The area between two curves

6D.1 The diagram shows the curve $y = x^2$ and the line $y = 9$. The enclosed region has been shaded. (i) Find the two points of intersection (labelled A and B). (ii) Using integration, show that the area of the shaded region is 36 square units.

6D.2 (i) Sketch the curves with equations $y = x^2 + 3$ and $y = 5 - x^2$ on the same axes, and shade the enclosed region. (ii) Find the co-ordinates of the points of intersection of the curves. (iii) Find the area of the shaded region.

6D.3 (i) Sketch the curve $y = x^3$ and the line $y = 4x$ on the same axes. (ii) Find the co-ordinates of the points of intersection of the curve $y = x^3$ and the line $y = 4x$. (iii) Find the total area of the region bounded by $y = x^3$ and $y = 4x$.

6D.4 (i) Sketch the curves with equations $y = x^2$ and $y = 4x - x^2$. (ii) Find the co-ordinates of the points of intersection of the curves. (iii) Find the area of the region enclosed by the curves.

6D.5 (i) Sketch the curves $y = x^2$ and $y = 8 - x^2$ and the line $y = 4$ on the same axes. (ii) Find the area of the region enclosed by the line $y = 4$ and the curve $y = x^2$. (iii) Find the area of the region enclosed by the line $y = 4$ and the curve $y = 8 - x^2$. (iv) Find the area enclosed by the curves $y = x^2$ and $y = 8 - x^2$.

6D.6 (i) Sketch the curve $y = x^2 - 6x$ and the line $y = -5$. (ii) Find the co-ordinates of the points of intersection of the line and the curve. (iii) Find the area of the region enclosed by the line and the curve.

6D.7 (i) Sketch the curve $y = x(4 - x)$ and the line $y = 2x - 3$. (ii) Find the co-ordinates of the points of intersection of the line and the curve. (iii) Find the area of the region enclosed by the line and the curve.

6D.8 Find the area of the region enclosed by the curves with equations $y = x^2 - 16$ and $y = 4x - x^2$.

6D.9 Find the area of the region enclosed by the curves with equations $y = -x^2 - 1$ and $y = -2x^2$.

6D.10 (i) Sketch the curve with equation $y = x^3 + 1$ and the line $y = 4x + 1$. (ii) Find the areas of the two regions enclosed by the line and the curve.

6D.11 The diagram shows the curve $y = 5x - x^2$ and the line $y = 4$. Find the area of the shaded region (the part of the region above the line $y = 4$ and below the curve).

6D.12 The diagram shows the curve with equation $y = x^2(3 - 2x - x^2)$. P and Q are points on the curve with co-ordinates $(-2, 12)$ and $(1, 0)$ respectively. (i) Find dy/dx. (ii) Find the equation of the line PQ. (iii) Prove that the line PQ is a tangent to the curve at both P and Q. (iv) Find the area of the region bounded by the line PQ and that part of the curve for which $-2 \leq x \leq 1$. [MEI]

6D.13 The diagram shows the graph of $y = 4x - x^3$. The point A has co-ordinates (2, 0). (i) Find dy/dx. Then find the equation of the tangent to the curve at A. (ii) The tangent at A meets the curve again at the point B. Show that the x co-ordinate of B satisfies the equation $x^3 - 12x + 16 = 0$. Find the co-ordinates of B. (iii) Calculate the area of the shaded region between the straight line AB and the curve. [MEI]

6D.14 The diagram shows the curve $y = (x - 2)^2$ and the line $y + 2x = 7$, which intersect at points A and B. Find the area of the shaded region (between the line and the curve). [Cambridge AS & A Level Mathematics 9709, Paper 12 Q9 June 2010]

6D.15 The diagram shows the curve $y = x^3 - 6x^2 + 9x$ for $x \geq 0$. The curve has a maximum point at A and a minimum point on the x axis at B. The normal to the curve at C(2, 2) meets the normal to the curve at B at the point D. The shaded region lies between the curve and the normal CD, bounded by the vertical line through B. (i) Find the co-ordinates of A and B. (ii) Find the equation of the normal to the curve at C. (iii) Find the area of the shaded region. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2009]

Exercise 6E · The area between a curve and the y axis

6E.1 Find the area of the region bounded by each of these curves, the y axis and the lines y = a and y = b. y = 3x + 1, a = 1, b = 7.

6E.2 $y = \sqrt{(x - 2)}$, $a = 0$, $b = 2$. (Find the area of the region bounded by this curve, the y axis and the lines $y = a$ and $y = b$.)

6E.3 $y = \sqrt[3]{x}$, $a = 0$, $b = 2$. (Find the area of the region bounded by this curve, the y axis and the lines $y = a$ and $y = b$.)

6E.4 $y = \sqrt{x} - 1$, $a = 0$, $b = 2$. (Find the area of the region bounded by this curve, the y axis and the lines $y = a$ and $y = b$.)

6E.5 $y = \sqrt[4]{x}$, $a = 1$, $b = 2$. (Find the area of the region bounded by this curve, the y axis and the lines $y = a$ and $y = b$.)

6E.6 $y = \sqrt[3]{x} - 2$, $a = -1$, $b = 1$. (Find the area of the region bounded by this curve, the y axis and the lines $y = a$ and $y = b$.)

Exercise 6F · The reverse chain rule

6F.1 Evaluate the following indefinite integrals. (i) $\int (x + 5)^4 \, dx$ · (ii) $\int (x + 7)^8 \, dx$ · (iii) $\int 1/(x - 2)^6 \, dx$ · (iv) $\int \sqrt{(x - 4)} \, dx$ · (v) $\int (3x - 1)^3 \, dx$ · (vi) $\int (5x - 2)^6 \, dx$ · (vii) $\int 3(2x - 4)^5 \, dx$ · (viii) $\int \sqrt{(4x - 2)} \, dx$ · (ix) $\int 4/(8 - x)^2 \, dx$ · (x) $\int 3/\sqrt{(2x - 1)} \, dx$

6F.2 Evaluate the following definite integrals. (i) $\int_1^5 \sqrt{(x-1)}\,dx$ · (ii) $\int_1^3 (x+1)^3\,dx$ · (iii) $\int_{-1}^4 (x-3)^4\,dx$ · (iv) $\int_0^3 (4-2x)^5\,dx$ · (v) $\int_5^9 \sqrt{(x-5)}\,dx$ · (vi) $\int_2^{10} \sqrt{(x-1)}\,dx$

6F.3 The graph of $y = (x-2)^3$ is shown (it passes through (2, 0) with a flat point there; the region between the curve, the x axis and the line $x = 4$ is shaded). (i) Evaluate $\int_2^4 (x-2)^3\,dx$. (ii) Without doing any calculations, state what you think the value of $\int_0^2 (x-2)^3\,dx$ would be. Give reasons. (iii) Confirm your answer by carrying out the integration.

6F.4 The graph of $y = (x - 1)^4 - 1$ is shown (it cuts the x axis at $x = 0$ and $x = 2$; region A lies above the x axis between $x = -1$ and $x = 0$, and region B lies below the x axis between $x = 0$ and $x = 2$). (i) Find the area of the shaded region A by evaluating $\int_{-1}^{0} ((x - 1)^4 - 1)\,dx$. (ii) Find the area of the shaded region B by evaluating an appropriate integral. (iii) Write down the area of the total shaded region. (iv) Why could you not just evaluate $\int_{-1}^{2} ((x - 1)^4 - 1)\,dx$ to find the total area?

6F.5 Find the area of the shaded region for each of the following graphs. (i) The graph of $y = (x - 3)^3$: the region is between the curve, the x axis and the line $x = 5$, starting at $x = 3$. (ii) The graph of $y = (x - 4)^2$: the region is between the curve, the x axis and the vertical line $x = 2$, ending at $x = 4$ where the curve touches the x axis.

6F.6 The equation of a curve is such that $dy/dx = 6/\sqrt{(3x - 2)}$. Given that the curve passes through the point P(2, 9), find (i) the equation of the normal to the curve at P, (ii) the equation of the curve.

6F.7 A curve is such that $dy/dx = 4/\sqrt{(6 - 2x)}$, and P(1, 8) is a point on the curve. (i) The normal to the curve at the point P meets the co-ordinate axes at Q and at R. Find the co-ordinates of the mid-point of QR. (ii) Find the equation of the curve. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q9 June 2006]

Exercise 6G · Improper integrals

6G.1 Evaluate the following improper integral. $\int_0^1 1/\sqrt{x}\, dx$

6G.2 Evaluate the following improper integral. $\int_{1}^{\infty} 1/x^3 \, dx$

6G.3 Evaluate the following improper integral. $\int_{1}^{\infty} 2/x^2 \, dx$

6G.4 Evaluate the following improper integral. $\int_{-\infty}^{-2} 2/x^3 \, dx$

6G.5 Evaluate the following improper integral. $\int_{1}^{\infty} -1/x^2 \, dx$

6G.6 Evaluate the following improper integral. $\int_0^4 6/\sqrt{x}\,dx$

Exercise 6H · Volumes of revolution

6H.1 Name six common objects which are solids of revolution.

6H.2 In each part of this question a region is defined in terms of the lines which form its boundaries. Draw a sketch of the region and find the volume of the solid obtained by rotating it through 360° about the x axis. (i) $y = 2x$, the x axis and the lines $x = 1$ and $x = 3$ · (ii) $y = x + 2$, the x axis, the y axis and the line $x = 2$ · (iii) $y = x^2 + 1$, the x axis and the lines $x = -1$ and $x = 1$ · (iv) $y = \sqrt{x}$, the x axis and the line $x = 4$

6H.3 (i) Sketch the line $4y = 3x$ for $x \geq 0$. (ii) Identify the area between this line and the x axis which, when rotated through 360° about the x axis, would give a cone of base radius 3 and height 4. (iii) Calculate the volume of the cone using (a) integration (b) a formula.

6H.4 In each part of this question a region is defined in terms of the lines which form its boundaries. Draw a sketch of the region and find the volume of the solid obtained by rotating through 360° about the y axis. (i) $y = 3x$, the y axis and the lines $y = 3$ and $y = 6$ · (ii) $y = x - 3$, the y axis, the x axis and the line $y = 6$ · (iii) $y = x^2 - 2$, the y axis and the line $y = 4$

6H.5 A mathematical model for a large garden pot is obtained by rotating through 360° about the y axis the part of the curve $y = 0.1x^2$ which is between $x = 10$ and $x = 25$ and then adding a flat base. Units are in centimetres. (i) Draw a sketch of the curve and shade in the cross-section of the pot, indicating which line will form its base. (ii) Garden compost is sold in litres. How many litres will be required to fill the pot to a depth of 45 cm? (Ignore the thickness of the pot.)

6H.6 The graph shows the curve $y = x^2 - 4$ (it cuts the x axis at $x = -2$ and $x = 2$ and the y axis at -4; the line $y = 12$ is also drawn). The region R is formed by the line $y = 12$, the x axis, the y axis and the curve $y = x^2 - 4$ for positive values of x. (i) Copy the sketch graph and shade the region R. The inside of a vase is formed by rotating the region R through 360° about the y axis. Each unit of x and y represents 2 cm. (ii) Write down an expression for the volume of revolution of the region R about the y axis. (iii) Find the capacity of the vase in litres. (iv) Show that when the vase is filled to 5/6 of its internal height it is three-quarters full. [MEI]

6H.7 The diagram shows the curve $y = 3x^{1/4}$. The shaded region is bounded by the curve, the x axis and the lines $x = 1$ and $x = 4$. Find the volume of the solid obtained when this shaded region is rotated completely about the x axis, giving your answer in terms of π. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q2 June 2007]

6H.8 The diagram shows part of the curve $y = a/x$, where a is a positive constant (the shaded region lies under the curve between $x = 1$ and $x = 3$). Given that the volume obtained when the shaded region is rotated through 360° about the x axis is 24π, find the value of a. [Cambridge AS & A Level Mathematics 9709, Paper 12 Q2 June 2010]

Check yourself

C.1 Find $\int(4x^3 - 6x^{-2} + 1/\sqrt{x})\,dx$.

C.2 A curve has gradient $dy/dx = 6x - 4$ and passes through (1, 3). Find its equation.

C.3 Find the exact area enclosed by $y = x^2$ and $y = 2x + 3$.

C.4 Find $\int_0^2 (3x - 1)^2 \, dx$.

C.5 The region under $y = \sqrt{x}$ between $x = 0$ and $x = 4$ is rotated through 360° about the x axis. Find the exact volume of the solid formed.

C.6 Evaluate $\int_{4}^{\infty} 3/x^2\,dx$.

Chapter quiz

Q1 Find $\int (6x^2 + 4x)\,dx$.

A	B	C	D
$2x^3 + 2x^2 + c$	$12x + 4 + c$	$6x^3 + 4x^2 + c$	$2x^3 + 4x^2 + c$

Q2 Find $\int x^{-3}\,dx$.

A	B	C	D
$-1/(2x^2) + c$	$-3x^{-4} + c$	$1/(2x^2) + c$	$-1/(3x^2) + c$

Q3 Find $\int \sqrt{x}\,dx$.

A	B	C	D
$(2/3)x^{3/2} + c$	$(3/2)x^{3/2} + c$	$1/(2\sqrt{x}) + c$	$(2/3)x^{1/2} + c$

Q4 A curve has $dy/dx = 2x + 1$ and passes through $(1, 5)$. What is its equation?

A	B	C	D
$y = x^2 + x + 3$	$y = x^2 + x + 5$	$y = x^2 + x + 2$	$y = 2x^2 + x + 2$

Q5 Evaluate $\int$ from 1 to 3 of $2x\,dx$.

A	B	C	D
8	9	6	4

Q6 Evaluate $\int$ from 0 to 2 of $(x^2 - 4)\,dx$.

A	B	C	D
−16/3	16/3	−8/3	−4/3

Q7 What is the area enclosed by $y = x$ and $y = x^2$?

A	B	C	D
1/6	1/3	1/2	1/12

Q8 Find $\int (2x + 1)^3\,dx$.

A	B	C	D
$(2x + 1)^4/8 + c$	$(2x + 1)^4/4 + c$	$(2x + 1)^4 + c$	$6(2x + 1)^2 + c$

Q9 Evaluate $\int$ from 1 to ∞ of $1/x^3\,dx$.

A	B	C	D
1/2	1	2	It has no finite value

Q10 Find $\int$ from 0 to 1 of $1/\sqrt{x}\,dx$.

A	B	C	D
2	1	1/2	It has no finite value

Q11 The region under $y = x$, from $x = 0$ to $x = 3$, is rotated about the x axis. What is the volume?

A	B	C	D
9π	3π	9π/2	27π

Q12 The region between $y = x^2$, the y axis and $y = 4$ is rotated about the y axis. What is the volume?

A	B	C	D
8π	16π	4π	64π/5

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p1-6-integration/