P1·5

Differentiation

CHAPTER ASSIGNMENT

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. Give exact answers (fractions, surds, π) unless a question asks for decimal places or significant figures. Unless stated otherwise, differentiate with respect to the variable on the right-hand side of the expression. Fractions are written a/b, with brackets where needed.

Practice questions from *Pure Mathematics 1* (Hodder Education, Sophie Goldie), Chapter 5 (Differentiation), Exercises 5A to 5H, the chapter's activities and its "Think about it" questions, for classroom use. Where a question needs a diagram, the diagram is described in words. The Check yourself and Chapter quiz questions are Guru Jazzy's own.

Exercise 5A · Differentiation from first principles

5A.1 Use the method in Example 5.1 (taking a chord joining P(x, y) to Q with x co-ordinate x + h and letting h → 0) to prove that the gradient of the curve $y = x^2$ at the point (x, y) is equal to 2x.

5A.2 Use the binomial theorem to expand $(x + h)^4$ and hence find the gradient of the curve $y = x^4$ at the point (x, y).

5A.3 Complete the table below, enter your answer to question 2, and suggest how the gradient pattern should continue when $f(x) = x^5$, $f(x) = x^6$ and $f(x) = x^n$ (where n is a positive whole number).

f(x)	f'(x) (gradient at (x, y))
x^2	$2x$
x^3	$3x^2$
x^4	
x^5	
x^6	
x^n	

5A.4 Prove the result when $f(x) = x^5$.

Activities in the text

5.1 Activity 5.1: On a curve, a tangent can be drawn by hand at a point A and its gradient estimated from a right-angled triangle (for example, y step / x step = 5.5 / 1.5 = 3.7). If you repeat the process at points B, C and D on a curve using different triangles, why do your answers differ slightly each time, and why is an algebraic method needed?

5.2 Activity 5.2: Take points X, Y, Z on the curve $y = x^2$ with x co-ordinates 3.1, 3.01 and 3.001 respectively, and find the gradients of the chords joining each of these points to (3, 9). What value are the gradients approaching?

5.3 Activity 5.3: Using the first-principles method (taking Q with x co-ordinate $x_0 + h$ on $y = x^2$), find the gradient of the tangent to the curve $y = x^2$ at (i) (1, 1) · (ii) (−2, 4) · (iii) (4, 16). What do you notice?

5.4 Think about it: How can you use the binomial theorem to prove Wallis's rule ($f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$) for positive integer values of n?

5.5 Activity 5.4: Plot the curve with equation $y = x^3 + 2$, for values of x from −2 to +2. On the same axes and for the same range of values of x, plot the curves $y = x^3 - 1$, $y = x^3$ and $y = x^3 + 1$. What do you notice about the gradients of this family of curves when x = 0? What about when x = 1 or x = −1?

5.6 Activity 5.5: Differentiate the equation $y = x^3 + c$, where c is a constant. How does this result help you to explain your findings in Activity 5.4?

5.7 Activity 5.6: Plot the graph of $y = x^4 - x^3 - 2x^2$, taking values of x from −2.5 to +2.5 in steps of 0.5, and answer these questions. (i) How many stationary points has the graph? (ii) What is the gradient at a stationary point? (iii) One of the stationary points is a maximum and the others are minima. Which are of each type? (iv) Is the maximum the highest point of the graph? (v) Do the two minima occur exactly at the points you plotted? (vi) Estimate the lowest value that y takes.

5.8 Activity 5.7: A cubic-like curve $y = f(x)$ has a local maximum at P and a local minimum at Q, and beneath it is the graph of $dy/dx = f'(x)$ (which crosses the x axis at P and Q, turning at a minimum between them). Sketch the graph of the gradient of dy/dx against x by extending vertical dashed lines through P and Q on to a third set of axes. What can you say about the gradient of dy/dx at P and at Q: is it positive, negative or zero?

5.9 Think about it: (1) A container holds liquid of volume V at depth h at time t. What information is given by dV/dh and dh/dt? What information is given by (dV/dh) × (dh/dt)? (2) A student differentiates $y = (x^2 - 2)^4$ by first multiplying out $(x^2 - 2)^4$ to get a polynomial of order 8. Prove that this heavy-handed method gives the same result as the chain rule, $dy/dx = 8x(x^2 - 2)^3$.

Exercise 5B · Differentiating standard functions, sums and differences

5B.1 Differentiate $y = x^5$ using the rules $y = kx^n \Rightarrow dy/dx = knx^{n-1}$ and $y = f(x) + g(x) \Rightarrow dy/dx = f'(x) + g'(x)$.

5B.2 Differentiate $y = 4x^2$.

5B.3 Differentiate $y = 2x^3$.

5B.4 Differentiate $y = x^{11}$.

5B.5 Differentiate $y = 4x^{10}$.

5B.6 Differentiate $y = 3x^5$.

5B.7 Differentiate $y = 7$.

5B.8 Differentiate $y = 7x$.

5B.9 Differentiate $y = 2x^3 + 3x^5$.

5B.10 Differentiate $y = x^7 - x^4$.

5B.11 Differentiate $y = x^2 + 1$.

5B.12 Differentiate $y = x^3 + 3x^2 + 3x + 1$.

5B.13 Differentiate $y = x^3 - 9$.

5B.14 Differentiate $y = (1/2)x^2 + x + 1$.

5B.15 Differentiate $y = 3x^2 + 6x + 6$.

5B.16 Differentiate $A = 4\pi r^2$.

5B.17 Differentiate $A = (4/3)\pi r^3$.

5B.18 Differentiate $d = (1/4)t^2$.

5B.19 Differentiate $C = 2\pi r$.

5B.20 Differentiate $V = l^3$.

5B.21 Differentiate $f(x) = x^{3/2}$.

5B.22 Differentiate $y = 1/x$.

5B.23 Differentiate $y = \sqrt{x}$.

5B.24 Differentiate $y = (1/5)x^{5/2}$.

5B.25 Differentiate $f(x) = 1/x^2$.

5B.26 Differentiate $f(x) = 5/x^3$.

5B.27 Differentiate $y = 2/\sqrt{x}$.

5B.28 Differentiate $f(x) = 4\sqrt{x} - 8/\sqrt{x}$.

5B.29 Differentiate $f(x) = x^{3/2} + x^{-3/2}$.

5B.30 Differentiate $f(x) = x^{5/3} - x^{-2/3}$.

5B.31 Differentiate $y = x(4x - 1)$.

5B.32 Differentiate $f(x) = (2x - 1)(x + 3)$.

5B.33 Differentiate $y = (x^2 + 6x)/x$.

5B.34 Differentiate $y = (4x^6 - 5x^4)/x^2$.

5B.35 Differentiate $y = x\sqrt{x}$.

5B.36 Differentiate $f(x) = 2x/\sqrt{x}$.

5B.37 Differentiate $g(x) = (3x^2 - 2x)/\sqrt{x}$.

5B.38 Differentiate $y = (x/4 + 4/x)(x^2 - x)$.

5B.39 Differentiate $h(x) = (\sqrt{x})^3$.

5B.40 Differentiate $y = (x^2 + 2x)(x - 4)/(2\sqrt{x})$.

Exercise 5C · Using differentiation to find gradients

5C.1 For each part of this question, (a) find dy/dx, (b) find the gradient of the curve at the given point. (i) $y = x^{-2}$; (0.25, 16) · (ii) $y = x^{-1} + x^{-4}$; (−1, 0) · (iii) $y = 4x^{-3} + 2x^{-5}$; (1, 6) · (iv) $y = 3x^4 - 4 - 8x^{-3}$; (2, 43) · (v) $y = \sqrt{x} + 3x$; (4, 14) · (vi) $y = 4x^{-1/2}$; (9, 1 1/3)

5C.2 (i) Sketch the curve $y = x^2 - 4$. (ii) Write down the co-ordinates of the points where the curve crosses the x axis. (iii) Differentiate $y = x^2 - 4$. (iv) Find the gradient of the curve at the points where it crosses the x axis.

5C.3 (i) Sketch the curve $y = x^2 - 6x$. (ii) Differentiate $y = x^2 - 6x$. (iii) Show that the point (3, −9) lies on the curve $y = x^2 - 6x$ and find the gradient of the curve at this point. (iv) Relate your answer to the shape of the curve.

5C.4 (i) Sketch, on the same axes, the graphs with equations $y = 2x + 5$ and $y = 4 - x^2$ for $-3 \leq x \leq 3$. (ii) Show that the point (−1, 3) lies on both graphs. (iii) Differentiate $y = 4 - x^2$ and so find its gradient at (−1, 3). (iv) Do you have sufficient evidence to decide whether the line $y = 2x + 5$ is a tangent to the curve $y = 4 - x^2$? (v) Is the line joining (2 1/2, 0) to (0, 5) a tangent to the curve $y = 4 - x^2$?

5C.5 The curve $y = x^3 - 6x^2 + 11x - 6$ cuts the x axis at $x = 1$, $x = 2$ and $x = 3$. (i) Sketch the curve. (ii) Differentiate $y = x^3 - 6x^2 + 11x - 6$. (iii) Show that the tangents to the curve at two of the points at which it cuts the x axis are parallel.

5C.6 (i) Sketch the curve $y = x^2 + 3x - 1$. (ii) Differentiate $y = x^2 + 3x - 1$. (iii) Find the co-ordinates of the point on the curve $y = x^2 + 3x - 1$ at which it is parallel to the line $y = 5x - 1$. (iv) Is the line $y = 5x - 1$ a tangent to the curve $y = x^2 + 3x - 1$? Give reasons for your answer.

5C.7 (i) Sketch, on the same axes, the curves with equations $y = x^2 - 9$ and $y = 9 - x^2$ for $-4 \leq x \leq 4$. (ii) Differentiate $y = x^2 - 9$. (iii) Find the gradient of $y = x^2 - 9$ at the points $(2, -5)$ and $(-2, -5)$. (iv) Find the gradient of the curve $y = 9 - x^2$ at the points $(2, 5)$ and $(-2, 5)$. (v) The tangents to $y = x^2 - 9$ at $(2, -5)$ and $(-2, -5)$, and those to $y = 9 - x^2$ at $(2, 5)$ and $(-2, 5)$ are drawn to form a quadrilateral. Describe this quadrilateral and give reasons for your answer.

5C.8 (i) Sketch, on the same axes, the curves with equations $y = x^2 - 1$ and $y = x^2 + 3$ for $-3 \leq x \leq 3$. (ii) Find the gradient of the curve $y = x^2 - 1$ at the point $(2, 3)$. (iii) Give two explanations, one involving geometry and the other involving calculus, as to why the gradient at the point $(2, 7)$ on the curve $y = x^2 + 3$ should have the same value as your answer to part (ii). (iv) Give the equation of another curve with the same gradient function as $y = x^2 - 1$.

5C.9 The function $f(x) = ax^3 + bx + 4$, where a and b are constants, goes through the point (2, 14) with gradient 21. (i) Using the fact that (2, 14) lies on the curve, find an equation involving a and b. (ii) Differentiate f(x) and, using the fact that the gradient is 21 when x = 2, form another equation involving a and b. (iii) By solving these two equations simultaneously find the values of a and b.

5C.10 In his book *Mathematician's Delight*, W.W. Sawyer observes that the arch of Victoria Falls Bridge appears to agree with the curve $y = (116 - 21x^2)/120$, taking the origin as the point mid-way between the feet of the arch, and taking the distance between its feet as 4.7 units (so the feet are at x = −2.35 and x = +2.35). (i) Find dy/dx. (ii) Evaluate dy/dx when x = −2.35 and when x = 2.35. (iii) Find the value of x for which dy/dx = −0.5.

5C.11 (i) Use your knowledge of the shape of the curve y = 1/x to sketch the curve y = 1/x + 2. (ii) Write down the co-ordinates of the point where the curve crosses the x axis. (iii) Differentiate y = 1/x + 2. (iv) Find the gradient of the curve at the point where it crosses the x axis.

5C.12 The sketch shows the graph of $y = 4/x^2 + x$ (for $x < 0$ it rises from the third quadrant, crosses the negative x axis and goes up towards the positive y axis; for $x > 0$ it falls from the positive y axis to a minimum in the first quadrant and then rises). (i) Differentiate $y = 4/x^2 + x$. (ii) Show that the point $(-2, -1)$ lies on the curve. (iii) Find the gradient of the curve at $(-2, -1)$. (iv) Show that the point $(2, 3)$ lies on the curve. (v) Find the gradient of the curve at $(2, 3)$. (vi) Relate your answer to part (v) to the shape of the curve.

5C.13 (i) Sketch, on the same axes, the graphs with equations $y = 1/x^2 + 1$ and $y = -16x + 13$ for $-3 \leq x \leq 3$. (ii) Show that the point $(0.5, 5)$ lies on both graphs. (iii) Differentiate $y = 1/x^2 + 1$ and find its gradient at $(0.5, 5)$. (iv) What can you deduce about the two graphs?

5C.14 (i) Sketch the curve $y = \sqrt{x}$ for $0 \leq x \leq 10$. (ii) Differentiate $y = \sqrt{x}$. (iii) Find the gradient of the curve at the point (9, 3).

5C.15 (i) Sketch the curve $y = 4/x^2$ for $-3 \leq x \leq 3$. (ii) Differentiate $y = 4/x^2$. (iii) Find the gradient of the curve at the point (−2, 1). (iv) Write down the gradient of the curve at the point (2, 1). Explain why your answer is −1 × your answer to part (iii).

5C.16 The sketch shows the curve $y = x/2 - 2/x$ for $x > 0$ (it rises from large negative y values near the y axis, crosses the positive x axis and continues rising). (i) Differentiate $y = x/2 - 2/x$. (ii) Find the gradient of the curve at the point where it crosses the x axis.

5C.17 The gradient of the curve $y = kx^{3/2}$ at the point $x = 9$ is 18. Find the value of k.

5C.18 Find the gradient of the curve $y = (x - 2)/\sqrt{x}$ at the point where $x = 4$.

Exercise 5D · Tangents and normals

5D.1 The graph of $y = 6x - x^2$ is a parabola through the origin and (6, 0). The marked point, P, is (1, 5). (i) Find the gradient function dy/dx. (ii) Find the gradient of the curve at P. (iii) Find the equation of the tangent at P.

5D.2 (i) Sketch the curve $y = 4x - x^2$. (ii) Differentiate $y = 4x - x^2$. (iii) Find the gradient of $y = 4x - x^2$ at the point (1, 3). (iv) Find the equation of the tangent to the curve $y = 4x - x^2$ at the point (1, 3).

5D.3 (i) Differentiate $y = x^3 - 4x^2$. (ii) Find the gradient of $y = x^3 - 4x^2$ at the point (2, −8). (iii) Find the equation of the tangent to the curve $y = x^3 - 4x^2$ at the point (2, −8). (iv) Find the co-ordinates of the other point at which this tangent meets the curve.

5D.4 (i) Sketch the curve $y = 6 - x^2$. (ii) Find the gradient of the curve at the points (−1, 5) and (1, 5). (iii) Find the equations of the tangents to the curve at these points. (iv) Find the co-ordinates of the point of intersection of these two tangents.

5D.5 (i) Sketch the curve $y = x^2 + 4$ and the straight line $y = 4x$ on the same axes. (ii) Show that both $y = x^2 + 4$ and $y = 4x$ pass through the point (2, 8). (iii) Show that $y = x^2 + 4$ and $y = 4x$ have the same gradient at (2, 8), and state what you conclude from this result and that in part (ii).

5D.6 (i) Find the equation of the tangent to the curve $y = 2x^3 - 15x^2 + 42x$ at (2, 40). (ii) Using your expression for dy/dx, find the co-ordinates of another point on the curve at which the tangent is parallel to the one at (2, 40). (iii) Find the equation of the normal at this point.

5D.7 (i) Given that $y = x^3 - 4x^2 + 5x - 2$, find dy/dx. The point P is on the curve and its x co-ordinate is 3. (ii) Calculate the y co-ordinate of P. (iii) Calculate the gradient at P. (iv) Find the equation of the tangent at P. (v) Find the equation of the normal at P. (vi) Find the values of x for which the curve has a gradient of 5. [MEI]

5D.8 (i) Sketch the curve whose equation is $y = x^2 - 3x + 2$ and state the co-ordinates of the points A and B where it crosses the x axis. (ii) Find the gradient of the curve at A and at B. (iii) Find the equations of the tangent and normal to the curve at both A and B. (iv) The tangent at A meets the tangent at B at the point P. The normal at A meets the normal at B at the point Q. What shape is the figure APBQ?

5D.9 (i) Find the points of intersection of $y = 2x^2 - 9x$ and $y = x - 8$. (ii) Find dy/dx for the curve and hence find the equation of the tangent to the curve at each of the points in part (i). (iii) Find the point of intersection of the two tangents. (iv) The two tangents from a point to a circle are always equal in length. Are the two tangents to the curve $y = 2x^2 - 9x$ (a parabola) from the point you found in part (iii) equal in length?

5D.10 The equation of a curve is $y = \sqrt{x}$. (i) Find the equation of the tangent to the curve at the point (1, 1). (ii) Find the equation of the normal to the curve at the point (1, 1). (iii) The tangent cuts the x axis at A and the normal cuts the x axis at B. Find the length of AB.

5D.11 The equation of a curve is $y = 1/x$. (i) Find the equation of the tangent to the curve at the point (2, 1/2). (ii) Find the equation of the normal to the curve at the point (2, 1/2). (iii) Find the area of the triangle formed by the tangent, the normal and the y axis.

5D.12 The sketch shows the graph of $y = \sqrt{x} - 1$ (starting at $(0, -1)$ and crossing the x axis at $(1, 0)$). (i) Differentiate $y = \sqrt{x} - 1$. (ii) Find the co-ordinates of the point on the curve $y = \sqrt{x} - 1$ at which the tangent is parallel to the line $y = 2x - 1$. (iii) Is the line $y = 2x - 1$ a tangent to the curve $y = \sqrt{x} - 1$? Give reasons for your answer.

5D.13 The equation of a curve is $y = \sqrt{x} - 1/(4x)$. (i) Find the equation of the tangent to the curve at the point where $x = 1/4$. (ii) Find the equation of the normal to the curve at the point where $x = 1/4$. (iii) Find the area of the triangle formed by the tangent, the normal and the x axis.

5D.14 The equation of a curve is $y = 9/\sqrt{x}$. The tangent to the curve at the point $(9, 3)$ meets the x axis at A and the y axis at B. Find the length of AB.

5D.15 The equation of a curve is $y = 2 + 8/x^2$. (i) Find the equation of the normal to the curve at the point (2, 4). (ii) Find the area of the triangle formed by the normal and the axes.

5D.16 The graph of $y = 3x - 1/x^2$ is shown (with two branches, one for $x < 0$ and one for $x > 0$). The point marked P on the right-hand branch is (1, 2). (i) Find the gradient function dy/dx. (ii) Use your answer from part (i) to find the gradient of the curve at P. (iii) Use your answer from part (ii), and the fact that the gradient of the curve at P is the same as that of the tangent at P, to find the equation of the tangent at P in the form $y = mx + c$.

5D.17 The graph of $y = x^2 + 1/x$ is shown. The point marked Q on the right-hand branch is (1, 2). (i) Find the gradient function dy/dx. (ii) Find the gradient of the tangent at Q. (iii) Show that the equation of the normal to the curve at Q can be written as $x + y = 3$. (iv) At what other points does the normal cut the curve?

5D.18 The equation of a curve is $y = x^{3/2}$. The tangent and normal to the curve at the point $x = 4$ intersect the x axis at A and B respectively. Calculate the length of AB.

5D.19 (i) The diagram shows the line $2y = x + 5$ and the curve $y = x^2 - 4x + 7$, which intersect at the points A and B (with A to the left of B). Find (a) the x co-ordinates of A and B, (b) the equation of the tangent to the curve at B, (c) the acute angle, in degrees correct to 1 decimal place, between this tangent and the line $2y = x + 5$. (ii) Determine the set of values of k for which the line $2y = x + k$ does not intersect the curve $y = x^2 - 4x + 7$. [Cambridge AS & A Level Mathematics 9709, Paper 12 Q10 November 2009]

5D.20 The equation of a curve is $y = 5 - 8/x$. (i) Show that the equation of the normal to the curve at the point P(2, 1) is $2y + x = 4$. This normal meets the curve again at the point Q. (ii) Find the co-ordinates of Q. (iii) Find the length of PQ. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q8 November 2008]

Exercise 5E · Maximum and minimum points, increasing and decreasing functions

5E.1 Given that $y = x^2 + 8x + 13$: (i) find dy/dx, and the value of x for which $dy/dx = 0$ · (ii) showing your working clearly, decide whether the point corresponding to this x value is a maximum or a minimum by considering the gradient either side of it · (iii) show that the corresponding y value is −3 · (iv) sketch the curve.

5E.2 Given that $y = x^2 + 5x + 2$: (i) find dy/dx, and the value of x for which $dy/dx = 0$ · (ii) classify the point that corresponds to this x value as a maximum or a minimum · (iii) find the corresponding y value · (iv) sketch the curve.

5E.3 Given that $y = x^3 - 12x + 2$: (i) find dy/dx, and the values of x for which $dy/dx = 0$ · (ii) classify the points that correspond to these x values · (iii) find the corresponding y values · (iv) sketch the curve.

5E.4 (i) Find the co-ordinates of the stationary points of the curve $y = x^3 - 6x^2$, and determine whether each one is a maximum or a minimum. (ii) Use this information to sketch the graph of $y = x^3 - 6x^2$.

5E.5 Find dy/dx when $y = x^3 - x$ and show that $y = x^3 - x$ is an increasing function for $x < -1/\sqrt{3}$ and $x > 1/\sqrt{3}$.

5E.6 Given that $y = x^3 + 4x$: (i) find dy/dx · (ii) show that $y = x^3 + 4x$ is an increasing function for all values of x.

5E.7 Given that $y = x^3 + 3x^2 - 9x + 6$: (i) find dy/dx and factorise the quadratic expression you obtain · (ii) write down the values of x for which $dy/dx = 0$ · (iii) show that one of the points corresponding to these x values is a minimum and the other a maximum · (iv) show that the corresponding y values are 1 and 33 respectively · (v) sketch the curve.

5E.8 Given that $y = 9x + 3x^2 - x^3$: (i) find dy/dx and factorise the quadratic expression you obtain · (ii) find the values of x for which the curve has stationary points, and classify these stationary points · (iii) find the corresponding y values · (iv) sketch the curve.

5E.9 (i) Find the co-ordinates and nature of each of the stationary points of $y = x^3 - 2x^2 - 4x + 3$. (ii) Sketch the curve.

5E.10 (i) Find the co-ordinates and nature of each of the stationary points of the curve with equation $y = x^4 + 4x^3 - 36x^2 + 300$. (ii) Sketch the curve.

5E.11 (i) Differentiate $y = x^3 + 3x$. (ii) What does this tell you about the number of stationary points of the curve with equation $y = x^3 + 3x$? (iii) Find the values of y corresponding to $x = -3, -2, -1, 0, 1, 2$ and 3. (iv) Hence sketch the curve and explain your answer to part (ii).

5E.12 You are given that $y = 2x^3 + 3x^2 - 72x + 130$. (i) Find dy/dx. P is the point on the curve where x = 4. (ii) Calculate the y co-ordinate of P. (iii) Calculate the gradient at P and hence find the equation of the tangent to the curve at P. (iv) There are two stationary points on the curve. Find their co-ordinates. [MEI]

5E.13 (i) Find the co-ordinates of the stationary points of the curve f(x) = 4x + 1/x. (ii) Find the set of values of x for which f(x) is an increasing function.

5E.14 The equation of a curve is $y = (1/6)(2x - 3)^3 - 4x$. (i) Find dy/dx. (ii) Find the equation of the tangent to the curve at the point where the curve intersects the y axis. (iii) Find the set of values of x for which $(1/6)(2x - 3)^3 - 4x$ is an increasing function of x. [Cambridge AS & A Level Mathematics 9709, Paper 12 Q10 June 2010]

5E.15 The equation of a curve is $y = x^2 - 3x + 4$. (i) Show that the whole of the curve lies above the x axis. (ii) Find the set of values of x for which $x^2 - 3x + 4$ is a decreasing function of x. The equation of a line is $y + 2x = k$, where k is a constant. (iii) In the case where $k = 6$, find the co-ordinates of the points of intersection of the line and the curve. (iv) Find the value of k for which the line is a tangent to the curve. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 June 2005]

5E.16 The equation of a curve C is $y = 2x^2 - 8x + 9$ and the equation of a line L is $x + y = 3$. (i) Find the x co-ordinates of the points of intersection of L and C. (ii) Show that one of these points is also the stationary point of C. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q4 June 2008]

Exercise 5F · The second derivative

5F.1 For each of the following functions, find dy/dx and d^2y/dx^2. (i) $y = x^3$ · (ii) $y = x^5$ · (iii) $y = 4x^2$ · (iv) $y = x^{-2}$ · (v) $y = x^{3/2}$ · (vi) $y = x^4 - 2/x^3$

5F.2 Find any stationary points on the curves of the following functions and identify their nature. (i) $y = x^2 + 2x + 4$ · (ii) $y = 6x - x^2$ · (iii) $y = x^3 - 3x$ · (iv) $y = 4x^5 - 5x^4$ · (v) $y = x^4 + x^3 - 2x^2 - 3x + 1$ · (vi) $y = x + 1/x$ · (vii) $y = 16x + 1/x^2$ · (viii) $y = x^3 + 12/x$ · (ix) $y = 6x - x^{3/2}$

5F.3 You are given that $y = x^4 - 8x^2$. (i) Find dy/dx. (ii) Find d^2y/dx^2. (iii) Find any stationary points and identify their nature. (iv) Hence sketch the curve.

5F.4 Given that $y = (x - 1)^2(x - 3)$: (i) multiply out the right-hand side and find dy/dx · (ii) find the position and nature of any stationary points · (iii) sketch the curve.

5F.5 Given that $y = x^2(x - 2)^2$: (i) multiply out the right-hand side and find dy/dx · (ii) find the position and nature of any stationary points · (iii) sketch the curve.

5F.6 The function $y = px^3 + qx^2$, where p and q are constants, has a stationary point at (1, −1). (i) Using the fact that (1, −1) lies on the curve, form an equation involving p and q. (ii) Differentiate y and, using the fact that (1, −1) is a stationary point, form another equation involving p and q. (iii) Solve these two equations simultaneously to find the values of p and q.

5F.7 You are given $f(x) = 4x^2 + 1/x$. (i) Find $f'(x)$ and $f''(x)$. (ii) Find the position and nature of any stationary points.

5F.8 For the function $y = x - 4\sqrt{x}$: (i) find dy/dx and d^2y/dx^2 · (ii) find the co-ordinates of the stationary point and determine its nature.

5F.9 The equation of a curve is $y = 6\sqrt{x} - x\sqrt{x}$. Find the x co-ordinate of the stationary point and show that the turning point is a maximum.

5F.10 For the curve $y = x^{5/2} - 10x^{3/2}$: (i) find the values of x for which $y = 0$ · (ii) show that there is a minimum turning point of the curve when $x = 6$ and calculate the y value of this minimum, giving the answer correct to 1 decimal place.

Exercise 5G · Practical maximum and minimum problems

5G.1 A farmer wants to construct a temporary rectangular enclosure of length x m and width y m for his prize bull while he works in the field. He has 120 m of fencing and wants to give the bull as much room to graze as possible. (i) Write down an expression for y in terms of x. (ii) Write down an expression in terms of x for the area, A, to be enclosed. (iii) Find dA/dx and d^2A/dx^2, and so find the dimensions of the enclosure that give the bull the maximum area in which to graze. State this maximum area.

5G.2 A square sheet of card of side 12 cm has four equal squares of side x cm cut from the corners. The sides are then turned up to make an open rectangular box to hold drawing pins. (i) Form an expression for the volume, V, of the box in terms of x. (ii) Find dV/dx and d^2V/dx^2, and show that the volume is a maximum when the depth is 2 cm.

5G.3 The sum of two numbers, x and y, is 8. (i) Write down an expression for y in terms of x. (ii) Write down an expression for S, the sum of the squares of these two numbers, in terms of x. (iii) By considering dS/dx and d^2S/dx^2, find the least value of the sum of their squares.

5G.4 A new children's slide is to be built with a rectangular open-top trough cross-section of base width y cm and two vertical sides of height x cm. A long strip of metal 80 cm wide is available for the chute and will be bent to form the base and two sides. The designer thinks that for maximum safety the area of the cross-section should be as large as possible. (i) Write down an equation linking x and y. (ii) Using your answer to part (i), form an expression for the cross-sectional area, A, in terms of x. (iii) By considering dA/dx and d^2A/dx^2, find the dimensions which make the slide as safe as possible.

5G.5 A carpenter wants to make a box to hold toys. The box is to be made so that its volume is as large as possible. A rectangular sheet of thin plywood measuring 1.5 m by 1 m is available to cut into pieces: two square end faces of side x m (stacked along the 1 m edge, with a third square x m by x m between them) and four equal rectangular side faces (extending across the remaining width of the 1.5 m sheet), while the remaining L-shaped shaded area is cut off and not used. (i) Write down the dimensions of one of the four rectangular faces in terms of x. (ii) Form an expression for the volume, V, of the made-up box, in terms of x. (iii) Find dV/dx and d^2V/dx^2. (iv) Hence find the dimensions of a box with maximum volume, and the corresponding volume.

5G.6 A piece of wire 16 cm long is cut into two pieces. One piece is 8x cm long and is bent to form a rectangle measuring 3x cm by x cm. The other piece is bent to form a square. (i) Find in terms of x: (a) the length of a side of the square, (b) the area of the square. (ii) Show that the combined area of the rectangle and the square is A cm^2 where $A = 7x^2 - 16x + 16$. (iii) Find the value of x for which A has its minimum value. (iv) Find the minimum value of A. [MEI]

5G.7 A piece of wire 30 cm long is going to be made into two frames for blowing bubbles. The wire is to be cut into two parts. One part is bent into a circle of radius r cm and the other part is bent into a square of side x cm. (i) Write down an expression for the perimeter of the circle in terms of r, and hence write down an expression for r in terms of x. (ii) Show that the combined area, A, of the two shapes can be written as $A = ((4 + \pi)x^2 - 60x + 225)/\pi$. (iii) Find the lengths that must be cut if the area is to be a minimum.

5G.8 A cylindrical can with a lid is to be made from a thin sheet of metal. Its height is to be h cm and its radius r cm. The surface area is to be 250π cm^2. (i) Find h in terms of r. (ii) Write down an expression for the volume, V, of the can in terms of r. (iii) Find dV/dr and d^2V/dr^2. (iv) Use your answers to part (iii) to show that the can's maximum possible volume is 1690 cm^3 (to 3 significant figures), and find the corresponding dimensions of the can.

5G.9 Charlie wants to add a rectangular extension with a floor area of 18 m^2 to the back of his house (so the house wall forms one side of length y m, and the three outside brick walls have lengths x m, y m and x m). He wants to use the minimum possible number of bricks, so he wants to know the smallest perimeter he can use. (i) Write down an expression for the area in terms of x and y. (ii) Write down an expression, in terms of x and y, for the total length, T, of the outside walls. (iii) Show that $T = 2x + 18/x$. (iv) Find dT/dx and d^2T/dx^2. (v) Find the dimensions of the extension that give a minimum value of T, and confirm that it is a minimum.

5G.10 A fish tank with a square base of side x m, height y m and no top is to be made from a thin sheet of toughened glass. (i) Write down an expression for the volume V in terms of x and y. (ii) Write down an expression for the total surface area A in terms of x and y. The tank needs a capacity of 0.5 m^3 and the manufacturer wishes to use the minimum possible amount of glass. (iii) Deduce an expression for A in terms of x only. (iv) Find dA/dx and d^2A/dx^2. (v) Find the values of x and y that use the smallest amount of glass and confirm that these give the minimum value.

5G.11 A closed rectangular box is made of thin card, and its length is three times its width. The height is h cm and the width is x cm (so the length is 3x cm). (i) The volume of the box is 972 cm^3. Use this to write down an expression for h in terms of x. (ii) Show that the surface area, A, can be written as $A = 6x^2 + 2592/x$. (iii) Find dA/dx and use it to find a stationary point. Find d^2A/dx^2 and use it to verify that the stationary point gives the minimum value of A. (iv) Hence find the minimum surface area and the corresponding dimensions of the box.

5G.12 A rectangular garden is planned with a central rectangular lawn of width x m and length y m (area 24 m^2) surrounded by a path of width 1 m along the top and bottom (length y m) edges and width 1.5 m along the left and right (width x m) edges. (i) Write down an expression for y in terms of x. (ii) Find an expression for the overall area of the garden, A, in terms of x. (iii) Find the smallest possible overall area for the garden.

5G.13 The diagram shows the cross-section of a hollow cone and a circular cylinder. The cone has radius 6 cm and height 12 cm, and the cylinder has radius r cm and height h cm. The cylinder just fits inside the cone with its base on the base of the cone and all of its upper edge touching the curved surface of the cone. (i) Express h in terms of r and hence show that the volume, V cm^3, of the cylinder is given by $V = 12\pi r^2 - 2\pi r^3$. (ii) Given that r varies, find the stationary value of V. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q5 November 2005]

Exercise 5H · The chain rule and connected rates of change

5H.1 Use the chain rule to differentiate the following functions. (i) $y = (x + 2)^3$ · (ii) $y = (2x + 3)^4$ · (iii) $y = (x^2 - 5)^3$ · (iv) $y = (x^3 + 4)^5$ · (v) $y = (3x + 2)^{-1}$ · (vi) $y = 1/(x^2 - 3)^3$ · (vii) $y = (x^2 - 1)^{3/2}$ · (viii) $y = (1/x + x)^3$ · (ix) $y = (\sqrt{x} - 1)^4$

5H.2 Given that $y = (3x - 5)^3$: (i) find dy/dx · (ii) find the equation of the tangent to the curve at (2, 1) · (iii) show that the equation of the normal to the curve at (1, −8) can be written in the form $36y + x + 287 = 0$.

5H.3 Given that $y = (2x - 1)^4$: (i) find dy/dx · (ii) find the co-ordinates of any stationary points and determine their nature · (iii) sketch the curve.

5H.4 Given that $y = (x^2 - x - 2)^4$: (i) find dy/dx · (ii) find the co-ordinates of any stationary points and determine their nature · (iii) sketch the curve.

5H.5 The length of a side of a square is increasing at a rate of 0.2 cm s^{-1}. At what rate is the area increasing when the length of the side is 10 cm?

5H.6 The force F newtons between two magnetic poles is given by the formula $F = 1/(500r^2)$, where r m is their distance apart. Find the rate of change of the force when the poles are 0.2 m apart and the distance between them is increasing at a rate of 0.03 m s^{-1}.

5H.7 The radius of a circular fungus is increasing at a uniform rate of 5 cm per day. At what rate is the area increasing when the radius is 1 m?

Check yourself

C.1 Differentiate $y = x^2 - 3x$ from first principles by finding the gradient of the chord from (x, y) to $(x + h, (x + h)^2 - 3(x + h))$ and letting $h \to 0$.

C.2 Differentiate $f(x) = (2x^3 - 5x + 4)/\sqrt{x}$ with respect to x.

C.3 Find the equations of the tangent and the normal to the curve $y = x^3 - 5x^2 + 8x$ at the point where $x = 2$.

C.4 Find the co-ordinates of the stationary points of the curve $y = 2x^3 - 3x^2 - 12x + 7$, and use the second derivative to determine the nature of each stationary point.

C.5 Differentiate $y = \sqrt{(3x^2 + 4)}$ with respect to x, and find the gradient of the curve at $x = 2$.

C.6 A spherical balloon of volume $V = (4/3)\pi r^3$ is being inflated so that its radius r cm increases at a constant rate of 2 cm s^{-1}. Find the exact rate at which the volume is increasing when $r = 5$ cm.

Chapter quiz

Q1 What is the gradient of the chord joining (2, 4) to (2 + h, (2 + h)²) on $y = x^2$?

A	B	C	D
4 + h	4 + 2h	2 + h	4h + h²

Q2 Differentiate $y = 3x^4 - 5x^2 + 7x - 9$ with respect to x.

A	B	C	D
$12x^3 - 10x + 7$	$12x^3 - 10x - 9$	$3x^3 - 5x + 7$	$12x^4 - 10x^2 + 7$

Q3 Find dy/dx when $y = 4\sqrt{x} - 6/x^2$.

A	B	C	D
$2/\sqrt{x} + 12/x^3$	$2/\sqrt{x} - 12/x^3$	$2\sqrt{x} + 12/x$	$4/(2\sqrt{x}) - 6/x^3$

Q4 What is the gradient of the curve $y = x^3 - 4x$ at the point (2, 0)?

A	B	C	D
8	4	12	0

Q5 Find the equation of the tangent to $y = x^2 - 3x + 4$ at the point (2, 2).

A	B	C	D
y = x	y = x + 2	y = −x + 4	y = 2x − 2

Q6 A curve has gradient 3 at (1, 4). What is the equation of the normal at (1, 4)?

A	B	C	D
x + 3y = 13	3x + y = 7	y = 3x + 1	x − 3y = −11

Q7 For what values of x is $f(x) = x^3 - 6x^2 + 5$ a decreasing function?

A	B	C	D
0 < x < 4	x < 0 or x > 4	x < 2	−4 < x < 0

Q8 Find the stationary points of $y = x^3 - 3x + 2$ and their nature.

A	B	C	D
Max at $(-1, 4)$, min at $(1, 0)$	Min at $(-1, 4)$, max at $(1, 0)$	Max at $(1, 0)$, min at $(-1, 4)$	Min at $(0, 2)$, max at $(1, 0)$

Q9 Given $y = 2x^4 - 3/x$, find the second derivative d^2y/dx^2.

A	B	C	D
$24x^2 - 6/x^3$	$8x^3 + 3/x^2$	$24x^2 + 6/x^3$	$24x^3 - 6/x^3$

Q10 Two positive numbers have sum 10. What is the maximum value of their product?

A	B	C	D
25	24	20	100

Q11 Differentiate $y = (2x^2 - 3)^5$ with respect to x.

A	B	C	D
$20x(2x^2 - 3)^4$	$5(2x^2 - 3)^4$	$10x(2x^2 - 3)^4$	$20x(4x)^4$

Q12 A circle's radius r cm increases at 3 cm s^{-1}. How fast is its area increasing when $r = 4$?

A	B	C	D
24π cm^2 s^{-1}	8π cm^2 s^{-1}	16π cm^2 s^{-1}	12π cm^2 s^{-1}

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p1-5-differentiation/