P1·4

Functions

CHAPTER ASSIGNMENT

Name _____ Class _____ No. _____ Date _____

Answer every question in the space given, showing all your working. Give exact answers unless a question asks for decimal places. Fractions are written a/b, with brackets where needed. $\mathbb{R}$ means all real numbers and $\mathbb{R}^+$ the positive real numbers. Where the book shows a picture, it is described in words or a table, and the question says so.

Practice questions from *Pure Mathematics 1* (Hodder Education, Sophie Goldie), Chapter 4 (Functions), Exercises 4A and 4B, and the transformation questions of Chapter 7 (Exercise 7F, questions 1 to 5), for classroom use. The Transformations of graphs section (not a chapter of the book, but part of the 9709 syllabus), the Check yourself and the Chapter quiz questions are Guru Jazzy's own.

Exercise 4A · The language of functions

4A.1 Describe each of the following mappings as either one-to-one, many-to-one, one-to-many or many-to-many, and say whether it represents a function. The book shows each as two ovals joined by arrows. Here the left oval has objects L1 to L4 (top to bottom) and the right oval has images R1, R2, R3 … (top to bottom). Table of arrows: (i) L1→R1, L2→R2, L3→R3, L4→R4 · (ii) L1→R1, L2→R1, L3→R2, L4→R4 (R3 is not used) · (iii) L1→R1 and R2, L2→R2, L3→R3, L4→R4 · (iv) L1→R1 and R2, L2→R3, L3→R4, L4→R5 · (v) L1→R1 and R3, L2→R2, L3→R3, L4→R3 (R4 and R5 are not used) · (vi) L1→R1, L2→R3, L3→R5, L4→R4 · (vii) L1→R1 and R2, L2→R2, L3→R2, R3 and R4, L4→R4 · (viii) L1, L2, L3 and L4 all →R2 (R1, R3, R4 are not used). (Read from the book's figure: check the picture if you have the book.)

4A.2 For each of the following mappings: (a) write down a few examples of inputs and corresponding outputs (b) state the type of mapping (one-to-one, many-to-one, etc.) (c) suggest a suitable domain. (i) Words → number of letters they contain · (ii) Side of a square in cm → its perimeter in cm · (iii) Natural numbers → the number of factors (including 1 and the number itself) · (iv) $x \to 2x - 5$ · (v) $x \to \sqrt{x}$ · (vi) The volume of a sphere in cm^3 → its radius in cm · (vii) The volume of a cylinder in cm^3 → its height in cm · (viii) The length of a side of a regular hexagon in cm → its area in cm^2 · (ix) $x \to x^2$

4A.3 (i) A function is defined by $f(x) = 2x - 5$, $x \in \mathbb{R}$. Write down the values of (a) f(0) (b) f(7) (c) f(−3). (ii) A function is defined by g: (polygons) → (number of sides). What are (a) g(triangle) (b) g(pentagon) (c) g(decagon)? (iii) The function t maps Celsius temperatures on to Fahrenheit temperatures. It is defined by $t: C \to 9C/5 + 32$, $C \in \mathbb{R}$. Find (a) t(0) (b) t(28) (c) t(−10) (d) the value of C when t(C) = C.

4A.4 Find the range of each of the following functions. (You may find it helpful to draw the graph first.) (i) $f(x) = 2 - 3x$, $x \geq 0$ · (ii) $f(\theta) = \sin\theta$, $0° \leq \theta \leq 180°$ · (iii) $y = x^2 + 2$, $x \in \{0, 1, 2, 3, 4\}$ · (iv) $y = \tan\theta$, $0° \leq \theta < 90°$ · (v) $f: x \to 3x - 5$, $x \in \mathbb{R}$ · (vi) $f: x \to 2^x$, $x \in \{-1, 0, 1, 2\}$ · (vii) $y = \cos x$, $-90° \leq x \leq 90°$ · (viii) $f: x \to x^3 - 4$, $x \in \mathbb{R}$ · (ix) $f(x) = 1/(1 + x^2)$, $x \in \mathbb{R}$ · (x) $f(x) = \sqrt{(x - 3)} + 3$, $x \geq 3$

4A.5 The mapping f is defined by $f(x) = x^2$ for $0 \leq x \leq 3$ and $f(x) = 3x$ for $3 \leq x \leq 10$. The mapping g is defined by $g(x) = x^2$ for $0 \leq x \leq 2$ and $g(x) = 3x$ for $2 \leq x \leq 10$. Explain why f is a function and g is not.

Exercise 4B · Composite and inverse functions

4B.1 The functions f, g and h are defined for $x \in \mathbb{R}$ by $f(x) = x^3$, $g(x) = 2x$ and $h(x) = x + 2$. Find each of the following, in terms of x. (i) fg (ii) gf (iii) fh (iv) hf (v) fgh (vi) ghf (vii) g^2 (viii) $(fh)^2$ (ix) h^2

4B.2 Find the inverses of the following functions. (i) $f(x) = 2x + 7$, $x \in \mathbb{R}$ · (ii) $f(x) = 4 - x$, $x \in \mathbb{R}$ · (iii) $f(x) = 4/(2 - x)$, $x \neq 2$ · (iv) $f(x) = x^2 - 3$, $x \geq 0$

4B.3 The function f is defined by $f(x) = (x - 2)^2 + 3$ for $x \geq 2$. (i) Sketch the graph of f(x). (ii) On the same axes, sketch the graph of $f^{-1}(x)$ without finding its equation.

4B.4 Express the following in terms of the functions f: $x \to \sqrt{x}$ and g: $x \to x + 4$ for $x > 0$. (i) $x \to \sqrt{(x + 4)}$ · (ii) $x \to x + 8$ · (iii) $x \to \sqrt{(x + 8)}$ · (iv) $x \to \sqrt{x} + 4$

4B.5 A function f is defined by f: $x \to 1/x$, $x \in \mathbb{R}$, $x \neq 0$. Find (i) $f^2(x)$ (ii) $f^3(x)$ (iii) $f^{-1}(x)$ (iv) $f^{999}(x)$.

4B.6 (i) Show that $x^2 + 4x + 7 = (x + 2)^2 + a$, where a is to be determined. (ii) Sketch the graph of $y = x^2 + 4x + 7$, giving the equation of its axis of symmetry and the co-ordinates of its vertex. The function f is defined by f: $x \to x^2 + 4x + 7$ with domain the set of all real numbers. (iii) Find the range of f. (iv) Explain, with reference to your sketch, why f has no inverse with its given domain. Suggest a domain for f for which it has an inverse. [MEI]

4B.7 The function f is defined by f: $x \to 4x^3 + 3$, $x \in \mathbb{R}$. Give the corresponding definition of f^{-1}. State the relationship between the graphs of f and f^{-1}. [UCLES]

4B.8 Two functions are defined for $x \in \mathbb{R}$ as $f(x) = x^2$ and $g(x) = x^2 + 4x - 1$. (i) Find a and b so that $g(x) = f(x + a) + b$. (ii) Show how the graph of $y = g(x)$ is related to the graph of $y = f(x)$ and sketch the graph of $y = g(x)$. (iii) State the range of the function g(x). (iv) State the least value of c so that g(x) is one-to-one for $x \geq c$. (v) With this restriction, sketch g(x) and $g^{-1}(x)$ on the same axes.

4B.9 The functions f and g are defined for $x \in \mathbb{R}$ by f: $x \to 4x - 2x^2$ and g: $x \to 5x + 3$. (i) Find the range of f. (ii) Find the value of the constant k for which the equation $gf(x) = k$ has equal roots. [Cambridge AS & A Level Mathematics 9709, Paper 12 Q3 June 2010]

4B.10 Functions f and g are defined by f: $x \to k - x$ for $x \in \mathbb{R}$, where k is a constant, and g: $x \to 9/(x + 2)$ for $x \in \mathbb{R}$, $x \neq -2$. (i) Find the values of k for which the equation $f(x) = g(x)$ has two equal roots and solve the equation $f(x) = g(x)$ in these cases. (ii) Solve the equation $fg(x) = 5$ when $k = 6$. (iii) Express $g^{-1}(x)$ in terms of x. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 June 2006]

4B.11 The function f is defined by f: $x \to 2x^2 - 8x + 11$ for $x \in \mathbb{R}$. (i) Express $f(x)$ in the form $a(x + b)^2 + c$, where a, b and c are constants. (ii) State the range of f. (iii) Explain why f does not have an inverse. The function g is defined by g: $x \to 2x^2 - 8x + 11$ for $x \geq A$, where A is a constant. (iv) State the largest value of A for which g has an inverse. (v) When A has this value, obtain an expression, in terms of x, for $g^{-1}(x)$ and state the range of g^{-1}. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q11 November 2007]

4B.12 The function f is defined by f: $x \to 3x - 2$ for $x \in \mathbb{R}$. (i) Sketch, in a single diagram, the graphs of $y = f(x)$ and $y = f^{-1}(x)$, making clear the relationship between the two graphs. The function g is defined by g: $x \to 6x - x^2$ for $x \in \mathbb{R}$. (ii) Express gf(x) in terms of x, and hence show that the maximum value of gf(x) is 9. The function h is defined by h: $x \to 6x - x^2$ for $x \geq 3$. (iii) Express $6x - x^2$ in the form $a - (x - b)^2$, where a and b are positive constants. (iv) Express $h^{-1}(x)$ in terms of x. [Cambridge AS & A Level Mathematics 9709, Paper 1 Q10 November 2008]

Transformations of graphs · book questions (Chapter 7, Exercise 7F)

7F.1 Starting with the graph of y = sin x, state the transformations which can be used to sketch each of the following curves. (i) y = sin (x − 90°) · (ii) y = sin 3x · (iii) 2y = sin x · (iv) y = sin (x/2) · (v) y = 2 + sin x

7F.2 Starting with the graph of y = cos x, state the transformations which can be used to sketch each of the following curves. (i) y = cos (x + 60°) · (ii) 3y = cos x · (iii) y = cos x + 1 · (iv) y = cos 2x

7F.3 For each of the following curves (a) sketch the curve (b) identify the curve as being the same as one of the following: y = ± sin x, y = ± cos x, or y = ± tan x. (i) y = sin (x + 360°) · (ii) y = sin (x + 90°) · (iii) y = tan (x − 180°) · (iv) y = cos (x − 90°) · (v) y = cos (x + 180°)

7F.4 Starting with the graph of y = tan x, find the equation of the graph and sketch the graph after the following transformations. (i) Translation of the column vector (0 across, 4 up), written with 0 on top and 4 underneath. (ii) Translation of the column vector (−30° across, 0 up), written with −30° on top and 0 underneath. (iii) One-way stretch with scale factor 2 parallel to the x axis.

7F.5 The graph of y = sin x is stretched with scale factor 4 parallel to the y axis. (i) State the equation of the new graph. (ii) Find the exact value of y on the new graph when x = 240°.

Transformations of graphs · Guru Jazzy's own questions

T.1 Describe fully the single transformation that maps the graph of y = f(x) on to the graph of (a) y = f(x) − 4 (b) y = f(x + 3) (c) y = 5f(x) (d) y = f(4x) (e) y = −f(x) (f) y = f(−x) (g) y = f(x/3)

T.2 The point (4, −3) lies on the curve y = f(x). Find the coordinates of its image on each of these curves. (a) y = f(x) + 5 (b) y = f(x − 2) (c) y = 2f(x) (d) y = f(3x) (e) y = −f(x) (f) y = f(−x) (g) y = 3f(x + 1) − 2

T.3 The graph of $y = x^2$ is transformed. Write down the equation of the new graph after (a) a translation of 3 units in the positive x direction (b) a translation of 5 units in the negative y direction (c) a stretch with scale factor 3 parallel to the y-axis (d) a stretch with scale factor 1/2 parallel to the x-axis (e) a reflection in the x-axis followed by a translation of 4 units up (f) a translation of 4 units up followed by a reflection in the x-axis.

T.4 Describe a sequence of transformations that maps the graph of $y = x^2$ on to the graph of (a) $y = (x + 2)^2 - 5$ (b) $y = 3(x - 1)^2 + 2$ (c) $y = x^2 + 6x + 4$ (complete the square first) (d) $y = 4x^2 - 1$.

T.5 The curve $y = f(x)$, where $f(x) = x^2 - 4x$, has its vertex at $(2, -4)$ and crosses the x-axis at $(0, 0)$ and $(4, 0)$. Write down the vertex and the x-intercepts of the curve (a) $y = f(x) + 3$ (b) $y = f(x - 1)$ (c) $y = f(2x)$ (d) $y = -f(x)$.

T.6 The graph of $y = \sin x$ is transformed into the graph of $y = 3 \sin 2x - 1$. (a) Describe the transformations, in a sensible order. (b) State the greatest and least values of y. (c) State the period of the new graph in degrees. (d) Sketch $y = 3 \sin 2x - 1$ for $0° \le x \le 360°$.

T.7 The curve $y = f(x)$ has a minimum point at $(2, 5)$. Write down the coordinates and the nature (maximum or minimum) of the turning point on each of these curves. (a) $y = f(x - 3) + 1$ (b) $y = -f(x)$ (c) $y = f(2x)$ (d) $y = 3f(-x)$

T.8 Let $f(x) = x^2 - 6x + 5$. (a) Express $f(x)$ in the form $(x - p)^2 + q$. (b) Describe the transformations that map $y = x^2$ on to $y = f(x)$. (c) Find the equation of the graph obtained by reflecting $y = f(x)$ in the y-axis, giving your answer in the form $y = x^2 + bx + c$.

Check yourself

C1 The functions f and g are defined for $x \in \mathbb{R}$ by $f(x) = 3x - 2$ and $g(x) = x^2 + 1$. (a) Find $fg(x)$ and $gf(x)$. (b) Solve the equation $fg(x) = gf(x)$, giving exact answers.

C2 The function f is defined by $f(x) = (2x + 1)/(x - 3)$, $x \neq 3$. Find $f^{-1}(x)$, and state the value that x cannot take in $f^{-1}(x)$.

C3 The function f is defined by $f(x) = x^2 - 6x + 4$ for $x \geq 3$. (a) Express $f(x)$ in the form $(x - a)^2 + b$. (b) State the range of f. (c) Find $f^{-1}(x)$ and state its domain. (d) Find $f^{-1}(4)$ and check it.

C4 The functions f and g are defined by $f(x) = 2x + 3$ and $g(x) = x - 1$, for $x \in \mathbb{R}$. Solve $gf(x) = f^{-1}(x)$.

C5 Say, with a reason, whether each function has an inverse for $x \in \mathbb{R}$. (a) $f(x) = x^3 + 1$ (b) $f(x) = x^2 - 4x$ (c) $f(x) = 5 - 2x$. Where there is no inverse, suggest a domain on which one exists.

C6 The point A(2, −1) lies on the curve $y = f(x)$. Find the coordinates of its image on the curve $y = 3 - 2f(x - 1)$, and describe the transformations in order.

Chapter quiz

Q1 $f(x) = 2x - 3$ and $g(x) = x^2$. What is $gf(x)$?

A	B	C	D
$2x^2 - 3$	$4x^2 - 9$	$(2x - 3)^2$	$x^2 - 3$

Q2 Which of these mappings is NOT a function?

A	B	C	D
$x \mapsto x^2, x \in \mathbb{R}$	$x \mapsto \cos x, x \in \mathbb{R}$	$x \mapsto \pm\sqrt{x}, x \geq 0$	$x \mapsto 2x + 1, x \in \mathbb{R}$

Q3 What is the largest real domain of $f(x) = \sqrt{(2x - 6)}$?

A	B	C	D
$x \geq 3$	$x \geq 6$	$x > 3$	$x \leq 3$

Q4 $f(x) = x^2 + 2$ for $x \geq 0$. What is $f^{-1}(x)$?

A	B	C	D
$\sqrt{x} - 2, x \geq 0$	$\sqrt{(x + 2)}, x \geq 2$	$\sqrt{(x - 2)}, x \geq 2$	$-\sqrt{(x - 2)}, x \geq 2$

Q5 What is the range of $f(x) = x^2 - 4x + 1$ for $x \in \mathbb{R}$?

A	B	C	D
$f(x) \geq 1$	$f(x) \geq -4$	$f(x) \geq 2$	$f(x) \geq -3$

Q6 The graph of $y = f^{-1}(x)$ is the graph of $y = f(x)$ reflected in which line?

A	B	C	D
$y = x$	the x-axis	the y-axis	$y = -x$

Q7 (3, 4) lies on $y = f(x)$. Where is its image on $y = f(x - 2) + 1$?

A	B	C	D
(1, 5)	(5, 5)	(5, 3)	(1, 3)

Q8 How is $y = f(2x)$ obtained from $y = f(x)$?

A	B	C	D
Stretch, scale factor 2, parallel to the x-axis	Stretch, scale factor ½, parallel to the x-axis	Stretch, scale factor 2, parallel to the y-axis	Translation of 2 units to the left

Q9 Which transformation maps $y = f(x)$ on to $y = -f(x)$?

A	B	C	D
Reflection in the y-axis	Reflection in the x-axis	Rotation of 180°	Reflection in $y = x$

Q10 $f(x) = x^2$ and $g(x) = x + 3$. What is $fg(2)$?

A	B	C	D
7	13	25	49

Q11 $g(x) = x^2 - 6x + 5$ for $x \geq c$. What is the least c for which g has an inverse?

A	B	C	D
−3	3	5	−4

Q12 $f(x) = 3x - 2$. What is $f^{-1}(7)$?

A	B	C	D
19	5	1/19	3

End of assignment · Check your answers against the chapter at gurujazzy.com/subjects/as-a-level-mathematics/p1-4-functions/